Special geometry on the moduli space of Calabi-Yau manifolds, Localization and Two-sphere partition functions.

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Introduction.

Abstract: The requirement the Space-time supersymmetry in the String theory is equivalent to the geometrical condition of the compactification 6 of 10 dimentions on Calabi-Yau(CY) threefold.

The properties of the Effective Lagrangian of the model, which describes the massless sector, are defined in terms of the so-called Special Kahler geometry on CY moduli space. I describe a new approach for computing the Special Kahler geometry based on the relation of Landau-Ginzburg superpotential of the model with a Frobenius manifold structure on the CY moduli space. I'l show how apply this approach for computing the Kahler metric on moduli space of the Calabi-Yau threefolds of Fer mat type. Also I show how the Kahler potentials are connected with the partition functions of the $N = (2, 2)$ Gauged Linear Sigma-Models on Two-Sphere.

Introduction.

The effective quantum field theory after compactification of the Superstring theory on a Calabi–Yau (CY) threefold X is defined in terms of so-called Special Kähler geometry of the CY moduli space.

The corresponding Kähler potential is given by the logarithm of the holomorphic volume of Calabi-Yau manifold X_ϕ:

$$G(\phi)_{a\bar{b}} = \partial_a \overline{\partial_b}\, K(\phi, \bar{\phi}),$$
$$e^{-K(\phi)} = \int_{X_\phi} \Omega \wedge \overline{\Omega},$$

where Ω is the holomorphic nonvanishing $3-0$ form on X_ϕ.
This can be rewritten in terms of periods of Ω as:

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$
$$e^{-K} = \omega_\mu(\phi)\, C_{\mu\nu}\, \overline{\omega_\nu(\phi)},$$

where $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is an intersection matrix of 3-cycles.

New approach

The computation of all periods is a very complicated problem and was done explicitly only in few examples.

In the talk I present the method to easily compute the Kähler metric for the large class of CY defined as hypersurfaces in weighted projective spaces.

The method uses a correspondence between the Hodge structure on the middle cohomology of CY manifold and the structure on the invariant Frobenius ring associated with the CY manifold.

This correspondence is realized by the oscillatory integral presentation for the periods of the holomorphic Calabi-Yau 3-form.

Clarifying this correspondence we obtain the efficient method for computing Special geometry on the CY moduli space.

Correspondence of the Hodge structure of $H^3(X)$ and R^Q.

Let X be a CY manifold realized as the zero locus of a quasi-homogeneous polynomial $W(x)$ in weighted P^4.

The key point of the approach is the correspondence between Cohomology $\mathbf{H^3(X)}$ with Hodge decomposition $\mathbf{H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)}$, the complex conjugation and Poincaré pairing the one side and the invariant ring $\mathbf{R^Q}$ defined by $\mathbf{W(x)}$ with its decomposition given by degree grading, an antiholomorphic involution M and the residue pairing $\eta_{\mu\lambda}$ on $\mathbf{R^Q}$ on the other side.

Using this correspondence we transform the formula for $K(\phi)$ to

$$e^{-K(\phi)} = \sigma_\mu(\phi)\,\eta_{\mu\lambda}\,M_{\lambda\nu}\,\overline{\sigma_\nu(\phi)}.$$

$\sigma_\mu(\phi)$ are periods that are in correspondence with the elements of the basis in R^Q presented by oscillatory integrals,

$\eta_{\mu\nu}$ is the residue pairing in the ring R^Q, $M_{\mu\nu}$ is the antiholomorphic involution of R^Q that is in correspondence with the complex cojugation $*$ in $H^3(X)$.

CY as the hypersurface in the weighted projective space

Let now $x_1, \dots, x_5$ be homogeneous coordinates in the weighted projective space $\mathbb{P}^4_{(k_1,\dots,k_5)}$, and let a Calabi-Yau X be defined as

$$X = \{x_1, \dots, x_5 \ \in \mathbb{P}^4_{(k_1,\dots,k_5)} |\ W_0(x) = 0\}$$

for a quasi-homogeneous polynomial $W_0(x)$

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i),$$

which defines the isolated singularity and

$$\deg W_0(x) = d = \sum_{i=1}^{5} k_i.$$

The last relation ensures that X is a CY manifold.
The moduli space of complex structures is then given by quasi-homogeneous deformations of this singularity:

$$W(x, \phi) = W_0(x) + \sum_{s=1}^{h_{21}} \phi_{\mathbf{s}} e_s(x),$$

where $e_s(x)$ are monomials of x which have the same degree d.

Hodge structure on middle cohomology

The holomorphic everywhere non-vanishing 3-form Ω is defined as

$$\Omega = \frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial W(x)/\partial x_4}$$

Periods of Ω are integrals over cycles of $H_3(X, \mathbb{R})$

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$

$H^3(X)$ possesses Hodge structure $H^3(X) = \oplus_{k=0}^3 H^{3-k,k}(X)$,

$\dim H^{3,0}(X) = \dim H^{0,3}(X) = 1, \ \dim H^{2,1}(X) = \dim H^{1,2}(X) = h^{2,1}$.

Poincaré pairing can be written through integrals over cycles q_μ as

$$< \chi_a, \chi_b >= \int_X \chi_a \wedge \chi_b = \int_{q_\mu} \chi_a \; C_{\mu\nu} \int_{q_\nu} \chi_b,$$

is invariant with respect to complex conjugation of (p, q)-forms.
$C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is the intersection matrix of 3-cycles.

Q-invariant Milnor ring

On the other hand the polynomial $W_0(x)$ defines a Milnor ring R_0. We consider its subring R^Q invariant with respect to the symmetry group Q that acts on $\mathbb{C}^5$ diagonally and preserves $W(x,\phi)$

$$R^Q = \left(\frac{\mathbb{C}[x_1,\cdots,x_5]}{\mathrm{Jac}(W_0)}\right)^Q, \quad \mathrm{Jac}(W_0) = \langle \partial_i W_0 \rangle_{i=1}^5.$$

R^Q becomes Frobenius ring if it is endowed with pairing

$$\eta(e_\alpha, e_\beta) = \mathrm{Res}\, \frac{e_\alpha(x) e_\beta(x)\, \mathrm{d}^5 x}{\prod_{i=1}^N \partial_i W_0(x)}.$$

$\dim R^Q = \dim H^3(X)$ and R^Q has the Hodge structure arising from the grading with degrees $0, d, 2d, 3d$

$$R^Q = (R^Q)^0 \oplus (R^Q)^1 \oplus (R^Q)^2 \oplus (R^Q)^3$$

. $\dim(R^Q)^0 = \dim(R^Q)^3 = 1, \quad \dim(R^Q)^1 = \dim(R^Q)^2 = h^{2,1}$

$Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_Q$

The next step is to introduce two differentials $D_\pm$

$$D_\pm = \mathrm{d} \pm \mathrm{d}W_0 \wedge, \qquad (D_\pm)^2 = 0$$

and two groups of $Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_Q$.
These groups inherit the grading from R^Q.
Choosing in the ring R^Q some basis $\{e_\mu(x)\}$ we will take $\{e_\mu(x)\,\mathrm{d}^5x\}$ as a basis of $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

These cohomology groups are in one-to-one correspondence with the middle cohomology group $\in H^3(X)$(Candelas 1988).

This isomorphism, defined below, maps the components $H^{3-q,q}(X)$ to the Hodge decomposition components of $H^5_\pm(\mathbb{C}^5)_Q$ spanned by $e_\mu(x)\,\mathrm{d}^5x$ with $e_\mu(x) \in (R^Q)^q$.

It also maps the Poincare pairing of differential forms to X to the pairing $\eta(e_\alpha, e_\beta)$ on the invariant ring R^Q.

Q-invariant relative homology and oscilatory integrals

Having $H^5_{D_\pm}(\mathbb{C}^5)_Q$ we define two Q-invariant relative homology groups $\mathcal{H}_5^{\pm,Q} := H_5(\mathbb{C}^5,\ \mathrm{Re} W_0 = L \to \pm\infty)_Q$ as a quotient of the relative homology group $H_5(\mathbb{C}^5,\ \mathrm{Re} W_0 = L \to \pm\infty)$.

For this purpose we define the pairing via oscillatory integrals

$$\langle Q^\pm_\mu, e_\nu(x)\,\mathrm{d}^5x\rangle := \int_{Q^\pm_\mu} e_\nu(x)\, e^{\mp W(x)}\mathrm{d}^5x.$$

We define the relative invariant homology groups $\mathcal{H}_5^{\pm,Q}$
as the quotient of $H_5(\mathbb{C}^5, W_0 = L,\ \mathrm{Re} L \to \pm\infty)$
by its subspace whose elements are orthogonal in repect to this pairing to all elements of Cohomology group $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

$H_3(X)$ versus R^Q correspondence

The crucial fact for what follows is that R^Q and $H^3(X)$ and all the additional structures on R^Q and $H^3(X)$ are isomorhic to each other.

We introduce an the isomorphism S

$$S(Q_\mu^+) = q_\mu, \quad Q_\mu^+ \in \mathcal{H}_5^{\pm,Q}, \;\; q_\mu \in H_3(X,\mathbb{Z})$$

defined as follows:
Let $\{q_\mu\}$ be a basis of $H_3(X,\mathbb{Z})$, then we choose the basis $Q_\mu^\pm$ of $\mathcal{H}_5^{\pm,Q}$ such that the following integrals over the corresponding cycles of these two bases are equal

$$\int_{q_\mu} \Omega_\phi = \int_{Q_\mu^\pm} e^{\mp W(x,\phi)}\, \mathrm{d}^5 x.$$

$H^3(X)$ versus $H^5_{D_\pm}(\mathbb{C}^5)_Q$ correspondence

Having isomorphism between $H_3(X)$ and $\mathcal{H}_5^{\pm,Q}$ we define the isomorphism between the two cohomology groups $H^3(X)$ and $H^5_{D_\pm}(\mathbb{C}^5)_Q$ also with help of oscilatory integrals.

Namely, take a basis of cycles $q_\mu \in H_3(X)$ and the corresponding to it basis of cycles $Q_\mu^\pm \in \mathcal{H}_5^{\pm,Q}$ at $\phi = 0$, then the form $\chi_\alpha \in H^3(X)$ corresponds to the form $e_\alpha(x)\,\mathrm{d}^5 x \in H^5_{D_\pm}(\mathbb{C}^5)_Q$ iff

$$\int_{q_\mu} \chi_\alpha = \int_{Q_\mu^\pm} e_\alpha(x)\, e^{\mp W_0(x)}\, \mathrm{d}^5 x$$

for all pairs $\{q_\mu, Q_\mu\}$.
So these two forms are isomorphic if they have the equal coordinates (i.e. periods) in some isomorphic bases.

This isomorphism preserves Hodge decomposition.
The pairing of the differential forms $\in H^3(X)$ and the pairing of the corresponding elements $\in R^Q$ coincide.

Coincidence of two pairings and the intersection matrix

We can rewrite the Poincaré pairing of $\chi_a, \chi_b \in H^3(X)$

$$< \chi_a, \chi_b >:= \int_X \chi_a \wedge \chi_b$$

as the bilinear expression of periods

$$< \chi_a, \chi_b >= \int_{q_\mu} \chi_a \; C_{\mu\nu} \int_{q_\nu} \chi_b,$$

where $C_{\mu\nu} = q_\mu \cap q_\nu$.

On the other hand the residue pairing $\eta(e_a, e_b)$ in ring R^Q can be written [Chiodo et al] in terms of periods and $\hat{C}_{\mu\nu} = Q^+_\mu \cap Q^-_\nu$ as

$$\eta(e_a, e_b) = \int_{Q^+_\mu} e_a e^{-W_0(x)} \mathrm{d}^5 x \; \hat{C}_{\mu\nu} \int_{Q^-_\mu} e_b e^{W_0(x)} \mathrm{d}^5 x,$$

Since $C_{\mu\nu} = \hat{C}_{\mu\nu}$ and $\int_{q_\mu} \chi_a = \int_{Q^\pm_\mu} e_a e^{\mp W_0(x)} \mathrm{d}^5 x$

we obtain the equality $< \chi_a, \chi_b >= \eta(e_a, e_b)$, which expresses the intersection matrix $C_{\mu\nu}$ in terms the pairing η_{ab}.

Anti-Involution on R^Q

The same isomorphism allows to define a the antiholomorphic involution M on the Q-invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$ that correspones to the complex conjugation $*$ of the forms $\in H^3(X)$. Let the form $\phi_\mu \in H^3(X)$ corresponds to $\{e_\mu(x)\} \in R^Q$ and let

$$*\phi_\mu = M_{\nu\mu}\phi_\nu.$$

The R^Q inherits this involution, and for the basis $\{e_\mu(x)\}$ the antiholomorphic operation $*$ reads as

$$* \, e_\mu(x) = M_{\nu\mu} e_\nu(x).$$

From this definition and since $(*)^2 = I$, it follows that $\bar{M}M = I$. It is convenient to introduce the basis $\Gamma^\pm_\mu \in \mathcal{H}^{\pm,Q}_5$ dual to the basis $\{e_\mu(x)\}$ such that:

$$\langle \Gamma^\pm_\mu, e_\nu(x)\, \mathrm{d}^5x \rangle = \int_{\Gamma^\pm_\mu} e_\nu(x)\, e^{\mp W_0(x)} \mathrm{d}^5x = \delta_{\mu\nu}.$$

This definition induces the antiholomorphic operation $*$ on $\Gamma^\pm_\mu$

$$*\Gamma^\pm_\mu = \bar{M}_{\mu\nu}\Gamma^\pm_\nu$$

How to compute $M_{\mu\nu}$

We define T as the transition matrix from cycles $\Gamma_\mu^\pm$ to any real cycles, say, Lefschetz thimbles $L_\mu^\pm = *L_\mu^\pm$

$$L_\mu^\pm = T_{\mu\nu}\Gamma_\nu^\pm.$$

Then we have

$$L_\mu^\pm = \overline{T}_{\mu\nu}\,(*\Gamma_\nu^\pm).$$

Comparing this relation with

$$*\Gamma_\mu^\pm = \bar{M}_{\mu\nu}\,\Gamma_\nu^\pm,$$

we obtain for M the useful expression in terms T

$$M = T^{-1}\,\bar{T}.$$

Obviously M does not depend on the choice of real cycles.
Using the definition $\langle \Gamma_\mu^\pm, e_\nu(x)\,\mathrm{d}^5x\rangle = \delta_{\mu\nu}$ we obtain the useful for computing $T_{\mu\nu}$ (and $M_{\mu\nu}$) relation

$$T_{\mu\nu} = \int_{L_\mu^\pm} e_\nu(x)\,e^{\mp W_0(x)}\mathrm{d}^5x.$$

Derivation the main formula for Kähler potential

We starts from the relation

$$e^{-K} = \omega_b^+(\phi)\ C_{ab}\ \overline{\omega_b^-(\phi)}.$$

We express the periods $\omega_a^\pm(\phi)$ given by oscilatory integrals over cycles $L_a^\pm$ in terms of $\sigma_\mu^\pm(\phi)$ given by oscilatory integrals over $\Gamma_\mu^\pm$

$$\omega_a^\pm(\phi) = \int_{L_a^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x = T_{a\mu}\sigma_\mu^\pm(\phi),$$

$$\sigma_\mu^\pm(\phi) = \int_{\Gamma_\mu^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

Also we use the expression for pairing on R^Q

$$\eta_{\mu\nu} = \int_{L_a^+} e_\mu e^{-W_0(x)} \mathrm{d}^5 x\ C_{ab} \int_{L_b^-} e_\nu e^{W_0(x)} \mathrm{d}^5 x = T_{a\mu} C_{ab} T_{b\nu}$$

Eliminating the matrix C_{ab} from these relations we obtain

$$e^{-K(\phi)} = \sum_{\mu,\nu,\lambda} \sigma_\mu^+(\phi)\ \eta_{\mu\lambda}\ M_{\lambda\nu}\ \overline{\sigma_\nu^-(\phi)}.$$

Example. Fermat threefolds

In this case CY manifold X is given by the equation

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} |\ W(x,\phi) = 0\}$$

$$W(x,\phi) = \sum_{i=1}^{5} x_i^{\frac{d}{k_i}} + \sum_{\mathbf{s}=\mathbf{1}}^{h_{21}} \phi_{\mathbf{s}} e_{\mathbf{s}}(x), \quad d = \sum_{i=1}^{5} k_i,$$

and $\frac{d}{k_i}$ are positive integers.

The monomials $e_{\mathbf{s}}(x) = e_{(s_1,\cdots,s_5)} := \prod_i x_i^{s_i}$ correspond to the deformation of the complex structure of X.
Their weights are equal to $\sum_{i=1}^{5} k_i s_i = d$ and each variable x_i has a non-negative integer power $s_i \leq \frac{d}{k_i} - 2$.
The number of such monomials is equal to the Hodge number h_{21}.

Q-invariant Ring

Considering $W_0(x)$ as an isolated singularity in $\mathbb{C}^5$ we have an associated Milnor ring

$$R_0 = \frac{\mathbb{C}[x_1, \cdots, x_5]}{\langle \partial_i W_0 \rangle}.$$

The bases of Milnor rings R_0 consist of monomials $e_\mu(x) = \prod_i x_i^{\mu_i}$. Each non-negative integer $\mu_i \leq \frac{d}{k_i} - 2$ and $\dim R_0 = \prod_i (\frac{d}{k_i} - 1)$.

Define the Z_d- invariant subring $R^Q \in R_0$ generated by the monomials $e_s(x)$ of weight d.
The basis of R^Q are elements $e_\mu(x) = e_{(\mu_1, \cdots, \mu_5)} := \prod_i x_i^{\mu_i}$ whose weights $\sum_{i=1}^5 k_i \mu_i$ are equal $0, d, 2d$ and $3d$.

The basis includes $e_\rho(x) := \prod_i x_i^{\frac{d}{k_i} - 2}$, $\rho = (\frac{d}{k_1} - 2, \cdots, \frac{d}{k_5} - 2)$.
The dimensions of the subspaces of degrees $0, d, 2d$ and $3d$ are $1, h_{21}, h_{21}$ and 1.
This grading defines a Hodge structure on R^Q which is isomorphic to the Hodge structure on $H^3(X)$.

Q-invariant ring

Fermat polynomials $W_0 = \sum_{i=1}^{5} x_i^{\frac{d}{k_i}}$ have a nice property that there is a symmetry group $\prod_i \mathbb{Z}_{d/k_i}$ that diagonally acts on $\mathbb{C}^5$:
$\alpha \cdot (x_1, \cdots, x_5) = (\alpha_1^{k_1} x_1, \cdots, \alpha_5^{k_5} x_5),\ \alpha_i^d = 1.$

The so–called quantum symmetry Q is the subgroup of $\prod_i \mathbb{Z}_{d/k_i}$ defined as follows.
Polynomial $W(x)$ is quasihomogeneous, therefore, in particular,
$W(\alpha^{k_1} x_1, \ldots, \alpha^{k_5} x_5) = W(x_1, \ldots, x_5)$, if $\alpha^d = 1$.
This acts trivially on the weighted projective space
as well as on X.
Thus in Fermat case $Q \simeq \mathbb{Z}_d$ and the subring R^Q is the Q-invariant part of the Milnor ring.

Phase symmetry, complex conjugation and pairing

The phase symmetry respects the Hodge decomposition.
The monomial basis $\{e_\mu(x) = e_{(\mu_1,\cdots,\mu_5)}(x) = \prod_i x^{\mu_i}\}$ of R^Q is an eigenbasis of the phase symmetry $\mathbb{Z}_d^5$, and each $e_\mu(x)$ has a unique weight. We can extend the phase symmetry action to the parameter space $\{\phi_s\}_{s=1}^{h_{2,1}}$ such that $W(x,\phi)$ is invariant under this action.
The complex conjugation acts on $H^3(X)$ as $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Due to the isomorphism between R^Q and $H^3(X)$ the complex conjugation M acts also on the elements of the ring R^Q as $*e_\mu(x) = p_\mu e_{\rho-\mu}(x)$, where p_μ is a constant.
On the invariant ring R^Q there exists the pairing turning it into a Frobenius algebra:

$$\eta_{\mu\nu} = \mathrm{Res}\,\frac{e_\mu(x)\,e_\nu(x)}{\prod_i \partial_i W_0(x)}.$$

For our monomial basis it is $\eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$.

Computation of periods $\sigma_\mu(\phi)$

To explicitly compute $\sigma^{\pm}_\mu(\phi)$, first we expand the exponent in the integral in ϕ representing $W(x,\phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma^{\pm}_\mu(\phi) = \sum_m \int_{\Gamma^{\pm}_\mu} \prod_r e_r(x)^{m_r}\, e^{\mp W_0(x)}\, \mathrm{d}^5 x\, \left(\prod_s \frac{(\pm\phi_s)^{m_s}}{m_s!} \right),$$

where $m := \{m_s\}_s,\ m_s \geq 0$ denotes a multi-index of powers of ϕ_s in the expansion above.
$\sigma^{-}_\mu(\phi) = (-1)^{|\mu|}\sigma^{+}_\mu(\phi)$, so we focus on $\sigma_\mu(\phi) := \sigma^{+}_\mu(\phi)$.

Each differential form $\prod_s e_s(x)^{m_s}\, \mathrm{d}^5 x$ belongs to $H^5_{D_\pm}(\mathbb{C}^5)_Q$.
It follows that it is equal to a linear combination of
$e_\mu d^5 x \in H^5_{D_\pm}(\mathbb{C}^5)_Q$ modulo D_+-exact terms.
We use this fact for computing oscilatory integrals taking into account that they vanish for D_+-exact terms and

$$\int_{\Gamma^+_\mu} e^{-W_0(x)} P(x)\, \mathrm{d}^5 x = \int_{\Gamma^+_\mu} e^{-W_0(x)} (P(x)\, \mathrm{d}^5 x + D_+ U)$$

for any polynomial $P(x)$ and any polynomial $4-$form U.

Computation of periods $\sigma_\mu(\phi)$

Let us denote $\sum_s m_s s_i = \nu_i + n_i \frac{d}{k_i},\ \nu_i < \frac{d}{k_i}$ (for later convenience). To compute

$$\int_{\Gamma_\mu^+} e^{-W_0(x)} \prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x,$$

we use the above property and the relation

$$\prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x =$$

$$= (-1) \left(n_1 - 1 + \frac{k_1(\nu_1 + 1)}{d} \right) x^{\nu_1 + (n_1 - 1)\frac{d}{k_1}} \prod_{i>1} x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x + D_+ U.$$

where

$$U = \frac{k_1}{d} x_1^{\nu_1 + 1 + (n_1 - 1)\frac{d}{k_1}} \prod_{i>1} x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_5.$$

Repeating this 4 times we obtain (modulo an exact term)

$$\prod_i x_i^{\nu_i+n_i\frac{d}{k_i}}\,\mathrm{d}^5x = (-1)\prod_i\left(n_i - 1 + \frac{k_i(\nu_i+1)}{d}\right)\prod_i x_i^{\nu_i+(n_i-1)\frac{d}{k_i}}\,\mathrm{d}^5x,$$

or

$$\prod_i x_i^{\nu_i+n_i\frac{d}{k_i}}\,\mathrm{d}^5x = (-1)^{\sum_i n_i}\prod_i\frac{\Gamma(\frac{k_i(\nu_i+1)}{d}+n_i)}{\Gamma(\frac{k_i(\nu_i+1)}{d})}\prod_i x_i^{\nu_i}\,\mathrm{d}^5x,\ \nu_i<\frac{d}{k_i}.$$

If any $\nu_i = \frac{d}{k_i}-1$, the form is exact, and the integral is zero.
Thus, the rhs is proportional to $e_\nu(x)\mathrm{d}^5x$.
Using the definition of Γ_μ^+ cycles $\int_{\Gamma_\mu^+} e_\nu(x)\,e^{-W_0(x)}\,\mathrm{d}^5x = \delta_{\mu\nu}$
we perform integrating over Γ_μ^+ and obtain that the period

$$\sigma_\mu(\phi) = \sum_{n_i\geq 0}\prod_i\frac{\Gamma\left(n_i+\frac{k_i(\mu_i+1)}{d}\right)}{\Gamma\left(\frac{k_i(\mu_i+1)}{d}\right)}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_s}}{m_s!},$$

where $\Sigma_n = \{m_s \mid \sum_s m_s s_i = \mu_i + \frac{d}{k_i}n_i\}$.

Formula for Kähler potential

Pick Lefschetz thimbles $L_\mu^\pm$ as a basis of cycles with real coefficients.
Let, as above, T as the transition matrix from cycles $\Gamma_\mu^\pm$ to Lefschetz thimbles $L_\mu^\pm$

$$\Gamma_\mu^\pm = (T^{-1})_{\mu\nu} \, L_\nu^\pm .$$

To compute the transition matrix $T_{\mu\nu}$ using the relation

$$T_{\mu\nu} = \int_{L_\mu^\pm} e_\nu(x) \, e^{\mp W_0(x)} \mathrm{d}^5 x.$$

Then we obtain

$$M = T^{-1} \, \bar{T}$$

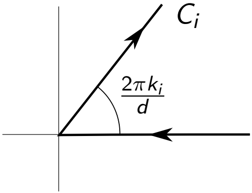
which we need to insert to the expression for Kähler potential together with $\eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$.

Lefschetz thimbles

Lefschetz thimbles $L^{\pm}_{\mu}$ are products of five one-dimensional cycles C_{μ_i}

$$L^{+}_{\mu} = \prod_{i=1}^{5} C_{\mu_i},$$

where $\sum_{i=1}^{5} k_i\mu_i$ are equal 0 modulo d
and $C_{\mu_i} = \hat{\rho}_i^{\mu_i} \cdot C_i$ with $\rho_i = e^{\frac{2\pi i k_i}{d}}$.
This definition of one-dimensional cycle C_{μ_i} means that this cycle is the path in x_i-plane obtained by rotating counter clockwise through angle $\frac{2\pi k_i \mu_i}{d}$ from the basic path C_i depicted on the figure



By construction $L^{\pm}_{\mu}$ are steepest descent/acsent cycles for $\mathrm{Re} W_0$.

Computing the matrices T and M

We now compute $T_{\alpha\mu}$ explicitly

$$T_{\alpha\mu} = \int_{L_\alpha^+} e_\mu \, e^{-W_0} \, \mathrm{d}^5 x = \rho^{(\bar{\alpha},\bar{\mu})} A(\mu),$$

where $\rho^{(\bar{\alpha},\bar{\mu})} = \prod_i e^{\frac{2\pi i k_i \mu_i}{d}}$ and A_μ is

$$A_\mu = \prod_i \left(\frac{k_i}{d}\right) \Gamma\left(\frac{k_i(\mu_i+1)}{d}\right).$$

We can show that $T^{-1}_{\bar{\mu}\bar{\alpha}} = B(\mu)[\bar{\rho}^{(\bar{\mu}+1,\bar{\alpha})} - 1]$,
where $B(\mu) = \prod_i \frac{1}{\Gamma\left(\frac{k_i(\mu_i+1)}{d}\right)}$.

Therefore

$$M_{\mu\nu} = (T^{-1}\bar{T})_{\mu\nu} = \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) \delta_{\mu,\rho-\nu},$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

Kähler potential for the moduli space of Fermat threefolds.

Substituting the explicit expressions for the periods σ_μ, the pairing $\eta_{\mu\nu}$, and the anti-involution M in the above expression for the Kähler potential on the moduli space, we obtain

$$e^{-K(\phi)} = \sum_\mu (-1)^{\deg(\mu)/d} \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) |\sigma_\mu(\phi)|^2,$$

where

$$\sigma_\mu(\phi) = \sum_{n_1,\ldots,n_5 \geq 0} \prod_{i=1}^{5} \frac{\Gamma(\frac{k_i(\mu_i+1)}{d} + n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!},$$

$0 \leq \mu_i \leq \frac{d}{k_i} - 2, \quad \sum_{i=1}^5 \mu_i = 0, d, 2d, 3d,$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Sigma_n = \{m_s \mid \sum_s m_s s_i = \mu_i + \frac{d}{k_i} n_i\}.$$

Kähler potential from exact partition function of GLSM.

Non-linear supersymmetric Calabi-Yau sigma-models admit ultraviolet description as $N = (2,2)$ gauged linear sigma-models (Witten). Consider a toric manifold $\mathbb{C}^N/(\mathbb{C}^*)^k$:

$$(x_1, \ldots, x_N) = (\lambda^{q_{i1}} x_1, \ldots, \lambda^{q_{iN}} x_N), \quad i = 1, \ldots, k$$

where $\lambda \in \mathbb{C}^*$ and k_{ij} is a $k \times N$ matrix. The hypersurface $\mathcal{Y}$ is given by any homogeneous polynomial $G(x_1, \ldots, x_N)$

$$G(\lambda^{q_{i1}} x_1, \ldots, \lambda^{q_{iN}} x_N) = \lambda^{d_i} G(x_1, \ldots, x_N), \quad i = 1, \ldots, k.$$

The CY condition is equivalent to the requirement

$$\sum_j k_{ij} + d_i = 0.$$

Let GLSM has a gauge group $\prod_j U(1)$ and $N+1$ chiral multiplets $(\Phi_1, \ldots, \Phi_N, P)$ with the $U(1)_j$ charges $Q_{ij} = (q_{1j}, \ldots, q_{Nj}, d_j)$.

Kähler potential from exact partition function of GLSM.

Any GLSM can be successfully placed on a sphere while preserving supersymmetry. The enough amount of supersymmetry allows one to compute the finite volume partition function of this theory exactly (Benini et all, Doroud et all)

$$Z = \sum_{m_l \in \mathbb{Z}} \prod_{l=1}^{k} e^{-i\theta_l m_l} \int_{\mathcal{C}} \prod_{l=1}^{k} \frac{d\tau_l}{(2\pi i)} e^{4\pi r_l \tau_l} \prod_{i=1}^{N+1} \frac{\Gamma\left(\sum_l Q_{il}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - \sum_l Q_{il}(\tau_l + \frac{m_l}{2})\right)},$$

It was conjectured and verified by a few explicit checks [Jockers et all] that Z yields the Kähler potential $K_K^{\mathcal{Y}}(\mathbf{z}, \bar{\mathbf{z}}) = -\log Z$ on the quantum Kähler moduli space for $\mathcal{Y}$ with Kähler parameters associated with complexified FI parameters

$$z_l = e^{2\pi r_l - i\theta_l}, \quad \bar{z}_l = e^{2\pi r_l + i\theta_l}.$$

Kähler potential from exact partition function of GLSM.

We use mirror symmetry to compare the exact partition function and $K_K^{\mathcal{Y}}(\mathbf{z}, \bar{\mathbf{z}})$. According to the mirror symmetry conjecture there exists a mirror manifold $\mathcal{X}$, such that

$$K_K^{\mathcal{Y}} = K_C^{\mathcal{X}}.$$

According to Batyrev, every CY manifold realized as a hypersurface in toric manifold has a mirror belonging to the same class. For the class of Fermat type hypersurfaces in the weighted projective spaces $P^4_{k_1,k_2,k_3,k_4,k_5}$ dual manifold has a matrix of charges of the following form

$$Q_{il} = \begin{cases} k_i s_{il}, & 1 \le i \le 5, \\ -d\delta_{i-5,l}, & 6 \le i \le h+5. \end{cases}$$

Kähler potential from exact partition function of GLSM.

The partition function in this case has the form

$$Z = \sum_{m_l \in \mathbb{Z}} \int_{\mathcal{C}} \prod_{l=1}^{h} \frac{d\tau_l}{(2\pi i)} \left(z_l^{\tau_l - \frac{m_l}{2}} \bar{z}_l^{\tau_l + \frac{m_l}{2}} \right) \times$$

$$\times \prod_{i=1}^{5} \frac{\Gamma\left(k_i \sum_l s_{il}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - k_i \sum_l s_{il}(\tau_l + \frac{m_l}{2})\right)} \prod_{l=1}^{h} \frac{\Gamma\left(-d(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 + d(\tau_l + \frac{m_l}{2})\right)}$$

For $r_l < 0$ each contour $\mathcal{C}$ can be closed to the right half-plane peaking up the poles at

$$d\left(\tau_l - \frac{m_l}{2}\right) = p_l; \quad p_l = 0, 1, \ldots \quad \text{such that} \quad p_l + m_l d < 0.$$

Kähler potential from exact partition function of GLSM.

It is convenient to introduce $\bar{p}_l = p_l + m_l d$. Then

$$Z = \pi^{-5} \sum_{p_l, \bar{p}_l} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{d}} \bar{z}_l^{-\frac{\bar{p}_l}{d}} \times$$

$$\times \prod_{i=1}^{5} \Gamma\left(\frac{k_i}{d} \sum_{l=1}^{h} s_{il} p_l\right) \Gamma\left(\frac{k_i}{d} \sum_{l=1}^{h} s_{il} \bar{p}_l\right) \sin\left(\frac{\pi k_i}{d} \sum_{l=1}^{h} s_{il} \bar{p}_l\right).$$

Each term, such that $k_i \sum_{l=1}^{h} s_{il} \bar{p}_l = 0 \mod d$, vanishes. It means that the sum effectively goes over the set

$$S_{\boldsymbol{\mu}} = \{p_l : \sum_{l=1}^{h} s_{il} p_l = \mu_i \mod d.\}$$

After simple algebra we find that

$$Z = \sum_{\boldsymbol{\mu}} (-1)^{|\boldsymbol{\mu}|} \prod_i \gamma\left(\frac{k_i(\mu_i + 1)}{d}\right) |\sigma_{\boldsymbol{\mu}}(\mathbf{z})|^2.$$