JKLMR conjecture and Batyrev’s polytopes.

Landau Institute

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Using the Batyrev mirror construction we study the mirror version of the relation proposed by Jockers et al.

between the exact partition functions of $N = (2, 2)$ gauged linear sigma-models on $S^2$ computed by Supersymmetric Localization and the Kähler potentials on the complex moduli spaces of the CY manifolds.
As it was shown by Candelas et al in order to obtain the space-time
supersymmetric effective theory in 4 dimensions it is necessary to
compactify the superstring theory on Calabi-Yau manifolds.

The dynamics of the massless sector of this theory is governed
by Kähler potentials of the moduli spaces of CY. Therefore for
studying the low energy dynamics one has to compute the CY
moduli geometry.

Taking into account this goal we study the mirror version of the
recently discovered by Jockers et al relation (JKLMR conjecture)
between the Kähler potentials on CY moduli space and the exact
partition functions of the $N = (2, 2)$ gauged linear sigma-models
(GLSM) on $S^2$ computed by Benini, Cremonesi and Doroud et al.
We will use the Batyrev mirror construction for finding the explicit correspondence between a GLSM with its gauge group and its set of the chiral superfields on the one hand side and the family of the Calabi-Yau manifolds $X$, defined as the hypersurfaces in the weighted projective spaces $\mathbb{P}^4_{k_1,k_2,k_3,k_4,k_5}$ on another hand side.

The key point of our approach is as follows. Let a CY hypersurface $X$ be given by zeros of the superpotential

$$W_X(x_1, x_2, x_3, x_4, x_5 | \psi_1, \ldots, \psi_h) = \sum_{a=1}^{h+5} C_a \prod_{i=1}^{5} x_i^{v_{ai}}, \sum_{i=1}^{5} k_i v_{ai} = \sum k_i.$$ 

Here $h$ is equal to the Hodge number $h_{21}$. The coefficients $C_a$ are some functions of the complex structure moduli $\psi_1, \ldots, \psi_h$. 

JKLMR conjecture and Batyrev's polytopes.
Main point. Mirror symmetry and toric geometry

In Batyrev approach the set of the exponents $v_{ai}$ corresponds to the lattice points of the Reflexive polytope $\Delta_X$. They are coordinates of the 5-vectors $\vec{V}_a \in R^5$, that is $v_{ai} = (\vec{V}_a)_i$. These vectors $\vec{V}_a$ being 5-dimensional are subjects to the linear relations

$$\sum_{a=1}^{h+5} Q_{al} \vec{V}_a = 0, \quad l = 1, \ldots, h.$$ 

The $Q_{al}$ is a set of integer numbers which corresponds to the relations space between the exponents of the monomials in $W_X$. These vectors $\vec{V}_a$ are the “edges” of the Fan that defines the toric manifold $\mathbb{C}^N/(\mathbb{C}^*)^h$, where $N = h + 5$. The symbol $\mathbb{C}^N/(\mathbb{C}^*)^h$ denotes the quotient $(\mathbb{C} - Z)/(\mathbb{C}^*)^h$, where $Z$ is a certain invariant subset.

This toric manifold can be defined by a set of the projective coordinates $y_1, \ldots, y_N \in \mathbb{C}^N$, which are subject to $h$ identifications

$$(y_1, \ldots, y_N) \sim (\lambda^{Q_{1l}}y_1, \ldots, \lambda^{Q_{Nl}}y_N), \quad l = 1, \ldots, h.$$
Then the Calabi-Yau manifold $Y$, which is the mirror of $X$, is realized as a surface in the toric manifold $\mathbb{C}^N // (\mathbb{C}^*)^h$ defined by the critical locus of a polynomial $W_Y(y_1, ..., y_N)$ which is weighted homogeneous with respect to all $h$ dilatations

$$W_Y(\lambda^{Q_1}y_1, \ldots, \lambda^{Q_N}y_N) = W_Y(y_1, \ldots, y_N), \quad l = 1, \ldots, h.$$  

The natural coordinates $(z_1, \ldots, z_{N-h})$ on the toric manifold $\mathbb{C}^N // (\mathbb{C}^*)^h$, invariant with respect to the group action, can be taken the monomials $z_j = \prod_{a=1}^N y_a^{v_{aj}}$. Thus $W_Y$ can be written as their linear combination

$$W_Y(y_1, \ldots, y_N) = \sum_{j=1}^5 \tilde{C}_j \prod_{a=1}^N y_a^{v_{aj}}$$

The manifold $Y$ given by this equation corresponds to the reflexive polytope $\Delta_Y$ that is polar dual to the polytope $\Delta_X$. 

Main point. Mirror symmetry and toric geometry
Calabi-Yau manifolds of such type arise as supersymmetric vacua manifold in 2d $N = (2, 2)$ gauged linear sigma-models (GLSM).

The weights $Q_{al}$ are just the charges of $N$ chiral superfields $(\Phi_1, \ldots, \Phi_N)$ under the gauge groups $U(1)_l$.

Thus, knowing the family of CY manifolds $X$ defined as the hypersurfaces a weighted projective by a polynomial $W_X$, we can find the corresponding $N = (2, 2)$ gauged linear sigma-model.

We find the charges $\{Q_{al}\}$ by solving the equation

$$
\sum_{a=1}^{h+5} Q_{al} \tilde{V}_a = 0, \quad l = 1, \ldots, h.
$$
Here we shortly recollect the connection between the
supersymmetric vacua in GLSM and the hypersurfaces in toric
manifolds.

Consider the $U(1)^h = \prod_{l=1}^{h} U(1)_l$ $N = (2, 2)$ gauge model $h$ vector
superfields and $N$ chiral matter fields $(\Phi_1, \ldots, \Phi_N)$ whose charges
under $U(1)_l$ are denoted by $(Q_{1l}, \ldots, Q_{Nl})$.

Lagrangian of the model also depends on coupling constants
$(e_1, \ldots, e_h)$, the real Fayet–Iliopoulos (FI) parameters $r_l$,
$l = 1, \ldots, k$, the theta angles and the superpotential $W_Y$. 
The potential term for the scalar fields in this Lagrangian is

\[ U(\phi) = \sum_{l=1}^{h} \frac{e_l^2}{2} \left( \sum_{a=1}^{N} Q_{al}|\phi_a|^2 - r_l \right)^2 + \frac{1}{4} \sum_{a=1}^{N} \left| \frac{\partial W_Y}{\partial \phi_a} \right|^2, \]

The supersymmetric ground states of the theory are parametrized by the minima of the potential modulo gauge equivalences.

\[ Y_r = \left\{ (\phi_1, \ldots \phi_N) \left| \sum_{a=1}^{N} Q_{al}|\phi_a|^2 = r_l, \quad l = 1, \ldots h, \quad \frac{\partial W_Y}{\partial \phi_a} = 0 \right\} / U(1)^h. \]

If we send all coupling constants \( e_l \) to infinity the model reduces to the \( N = (2, 2) \) non-linear sigma model with the target space \( Y_r \).

The toric manifolds \( \mathbb{C}^N // (\mathbb{C}^*)^h \) and \( Y_r \) are equivalent.

This establishes a one-to-one correspondence between the hypersurfaces \( Y \) in toric manifolds and the gauged linear sigma-models (GLSM).

JKLMR conjecture and Batyrev’s polytopes.
It was shown by Benini et al and Doroud et al that GLSM can be put on the 2-sphere while preserving the $N = (2, 2)$ supersymmetry, which allows one to compute the partition function of this theory exactly, using the Supersymmetric Localization

$$Z_Y = \sum_{m_l \in \mathbb{Z}} \prod_{l=1}^{h} e^{-i\theta_l m_l} \times$$

$$\times \int_{C_1} \cdots \int_{C_k} \prod_{l=1}^{h} \frac{d\tau_l}{(2\pi i)} e^{4\pi r_l \tau_l} \prod_{a=1}^{N} \frac{\Gamma\left(q_a/2 + \sum_{l=1}^{k} Q_{al}(\tau_l - \frac{m_l}{2})\right)}{\Gamma\left(1 - q_a/2 - \sum_{l=1}^{k} Q_{al}(\tau_l + \frac{m_l}{2})\right)},$$

where the contours $C_l$ go along the imaginary axis. The partition function does not depend either on the coupling constants $(e_1, \ldots, e_h)$ or on the details of the polynomial $W_Y$. 
Thus effectively $Z_Y$ picks up only the contribution of the massless fields and gives the exact partition function of the non-linear sigma-model.

It was conjectured by Jockers et al. that the exact expression for the partition function coincides with the Kähler potential $K^Y_K = - \log Z_Y$ on the quantum Kähler moduli space for Calabi-Yau manifold $Y$.

This conjecture was verified for a few examples. The problem with this check is the lack of simple definition of the function $K^Y_K$.

Here we will adopt a different point of view based on mirror symmetry and Batyrev approach to it.
Mirror version of JKLMR conjecture

The mirror symmetry implies an equality

\[ K^Y_K = K^X_C, \quad e^{-K^X_C} = -i \int_X \Omega \wedge \bar{\Omega}, \]

where \( K^X_C \) is the Kähler potential for complex structure moduli of the mirror manifold \( X \).

The last potential has a transparent definition in terms of integral of the uniquely defined holomorphic form \( \Omega \) on \( X \).

The manifold \( X \) is canonically related to the manifold \( Y \) realized as a hypersurface in toric manifold by the mirror map.

So the conjecture of Jockers et al can be reformulated as

\[ \int_X \Omega \wedge \bar{\Omega} = i Z_Y. \]

We will call this the mirror version of JKLMR conjecture.
Batyrev construction

Consider the example of Quintic threefold. In this case the CY $X$ is defined as the hypersurface in the projective space $\mathbb{P}^4$, that is a set of five complex coordinates $(x_1, \ldots, x_5)$ identified by

$$(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5).$$

The hypersurface $X \in \mathbb{P}^4$ is given by the zeroes of the superpotential

$$W_X(x_1, x_2, x_3, x_4, x_5 | \psi_1, \ldots, \psi_h) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{h} \psi_l e_l(x),$$

$$e_l(x) = \prod_{i=1}^{5} x_i^{s_{li}}, \quad \text{with} \quad 0 \leq s_{li} \leq 3, \quad h = 101 \quad \text{and} \quad \sum_{i=1}^{5} s_{li} = 5.$$

The parameters $\psi_l$ represent deformations of the complex structure of the manifold $X$. 
Batyrev construction

The manifold $Y$, the mirror to $X$, can be realized as a hypersurface in the toric manifold defined as the quotient $\mathbb{C}^{h+5} \big/ \big( \mathbb{C}^* \big)^h$. The action of the group $\big( \mathbb{C}^* \big)^h$ depend of by the matrix $Q_{al}$, as was explained above. Batyrev’s construction tells us how to find this matrix. The vectors $V_a$ whose components are the exponents $v_{ai}$ in $W_X$ equal

$$v_{ai} = \begin{cases} 5\delta_{a,i}, & 1 \leq a \leq 5, \\ sa_{-5,i}, & 6 \leq a \leq h + 5. \end{cases}$$

constitute the Fan of dual manifold $Y$.

As it was explained above one can find the charges $Q_{al}$ of the chiral superfields in the corresponding GLSM by finding the linear dependence between these vectors

$$\sum_{a=1}^{h+5} Q_{al} \cdot V_a = 0.$$
The convenient choice of solutions for this equation is then

\[ Q_{al} = \begin{cases} 
    s_{la}, & 1 \leq a \leq 5, \\
    -5 \delta_{a-5,l}, & 6 \leq a \leq 106. 
\end{cases} \]

We can see that \( \sum_{a=1}^{106} Q_{al} = 0 \).

Therefore, as explained by Witten the FI parameters and the theta parameters do not run with RG flow and the \( U(1) \) axial symmetry is unbroken. The FI and theta parameters \( r_l \) and \( \theta_l \) remain arbitrary parameters also in the quantum theory.

The superpotential \( W_Y \) can be written in terms of the invariant coordinates as it is presented above.

It is convenient to introduce the new denotations for the 106 chiral matter fields (\( \Phi_1, \ldots, \Phi_{106} \)) and their scalar components, whose charges under \( U(1)_l \) we found, as follows

\[ \Phi_a = \begin{cases} 
    S_a, & 1 \leq a \leq 5, \\
    P_{a-5}, & 6 \leq a \leq 106. 
\end{cases} \]
The GLSM

The field $P_1$ corresponds to $\nu_{6i}$, whose components $\nu_{6i} = 1$ will play a somehow distinguished role. The superpotential $W_Y$ in the considered case can be written as

$$W_Y = P_1 \cdot G(S_1, \ldots, S_5; P_2, \ldots, P_{101}).$$

The potential term for the scalar fields in the Lagrangian, which was given above, will then take the form

$$U(\phi) = \sum_{l=1}^{101} \frac{e_l^2}{2} \left( \sum_{a=1}^{5} s_{al} |S_a|^2 - 5 |P_l|^2 - r_l \right)^2 + \frac{1}{4} |G(S_1, \ldots, S_5; P_2, \ldots, P_{101})|^2 + \frac{1}{4} |P_1|^2 \sum_{a=1}^{5} \left| \frac{\partial G}{\partial S_a} \right|^2 + \frac{1}{4} |P_1|^2 \sum_{l=2}^{101} \left| \frac{\partial G}{\partial P_l} \right|^2.$$
Depending on the values of FI parameters in the theory the different phases are expected to occur. In a suitable region of $r_l$ the vacuum manifold is a set of $(S_i, P_l)$ obeying the equations

$$\sum_{a=1}^5 s_{al} |S_a|^2 - 5 |P_l|^2 - r_l = 0, \quad G(S_1, \ldots, S_5; P_2, \ldots, P_{101}) = 0, \quad P_1 = 0$$

divided by the gauge group action.

This is the explicit definition of the hypersurface $Y$ in the toric space $Y_r$, which is the vacuum manifold of the model.
In the case considered the exact partition function $Z_Y$ can be written in the form

$$Z_Y = \sum_{m_l} \int_{C_1} \cdots \int_{C_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left( z_l^{-\tau_l + \frac{m_l}{2}} \bar{z}_l^{-\tau_l - \frac{m_l}{2}} \right) \times$$

$$\frac{\Gamma(1 - 5(\tau_1 - \frac{m_1}{2}))}{\Gamma(5(\tau_1 + \frac{m_1}{2}))} \prod_{a=1}^{5} \frac{\Gamma\left( \sum_l s_{al}(\tau_l - \frac{m_l}{2}) \right)}{\Gamma\left(1 - \sum_l s_{al}(\tau_l + \frac{m_l}{2}) \right)} \prod_{l=2}^{101} \frac{\Gamma(-5(\tau_l - \frac{m_l}{2}))}{\Gamma(1 + 5(\tau_l + \frac{m_l}{2}))},$$

where

$$z_l = e^{-(2\pi r_l + i\theta_l)}.$$

The contours $C$ go slightly to the left of the imaginary axes: $\tau_l = -\epsilon + it_l$. We consider the expansion of this integral for large values of $|z_l|$, that is for $r_l < 0$. 
Check of JKLMR conjecture

For \( r_l < 0 \) each contour \( C_{\uparrow} \) can be closed to the right half-plane picking up the poles at

\[
5 \left( \tau_l - \frac{m_l}{2} \right) = p_l; \quad p_1 = 1, 2, \ldots, \quad p_l = 0, 1 : \quad p_l + 5m_l > 0.
\]

It is convenient to introduce \( \bar{p}_l = p_l + 5m_l \) which varies from 0 to \( \infty \). Then the partition function can be rewritten as

\[
Z = \pi^{-5} \sum_{\bar{p}_l \geq 0} \sum_{p_l \geq 0} \prod_l \frac{(-1)^{p_l}}{p_l! \bar{p}_l!} z_l^{-\frac{p_l}{5}} \bar{z}_l^{-\frac{\bar{p}_l}{5}} \times
\]

\[
\times \prod_{i=1}^{5} \Gamma \left( \frac{1}{5} \sum_{l=1}^{h} s_{li} \bar{p}_l \right) \Gamma \left( \frac{1}{5} \sum_{l=1}^{h} s_{li} \bar{p}_l \right) \sin \left( \frac{\pi}{5} \sum_{l=1}^{h} s_{li} \bar{p}_l \right).
\]

Where the sum over \( \bar{p}_l \) goes over the set which is restricted by the condition \( \sum s_{li} \bar{p}_l = \sum s_{li} p_l \mod 5 \) and the terms such that \( \sum_{l=1}^{h} s_{il} \bar{p}_l = 0 \mod 5 \) are removed.
Check of JKLMR conjecture

It means that the sum effectively goes over

$$S_{\mu,n} = \left\{ p_l : \sum_{l=1}^{h} s_{ll} p_l = \mu_i + 5n_i, \ 1 \leq \mu_i \leq 4, \ 0 \leq n_i \right\}.$$ 

Obviously, if \(\{p_l\}\) belongs to \(S_\mu\) so does the set \(\{\bar{p}_l\}\). Using,

$$\prod_{i=1}^{5} \sin \left( \frac{\pi}{5} \sum_{l=1}^{h} s_{ll} \bar{p}_l \right) = (-1)^{|\mu|} \prod_{i=1}^{5} \sin \left( \frac{\pi \mu_i}{5} \right) \prod_{l=1}^{h} (-1)^{\bar{p}_l},$$

we find that

$$Z = \sum_{\mu} (-1)^{|\mu|} \prod_{i=1}^{5} \frac{\Gamma \left( \frac{\mu_i}{5} \right)}{\Gamma \left( 1 - \frac{\mu_i}{5} \right)} |\sigma_{\mu}(z)|^2,$$

$$\sigma_{\mu}(z) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma \left( \frac{\mu_i}{5} + n_i \right)}{\Gamma \left( \frac{\mu_i}{5} \right)} \sum_{p \in S_{\mu,n}} \prod_{l=1}^{h} \frac{(-1)^{p_l} z_{l}^{-\frac{p_l}{5}}}{p_l!}.$$ 

We find that \(Z\) exactly coincides with the expression for \(\int_{X} \Omega \wedge \bar{\Omega}\) obtained by us earlier, if we identify the parameters as \(z_l = \psi_l^{-5}\).
Thus, we have shown how, starting from the superpotential $W_X(x)$ which defines a CY manifold $X$, to find $N = (2, 2)$ gauged linear sigma model whose vacuum manifold $Y$ is the mirror to $X$. We have also verified here the conjecture by Jockers et al for the quintic threefold.

Actually, in the region $r_l < 0$, where the formula for the metric on the Complex moduli space of the CY manifold $X$ was obtained, the theory is expected to describe the Landau-Ginzburg phase.

On the other hand, in order to get a metric on the Kähler moduli space of the manifold $Y$ which is mirror to the $X$ one need to perform an analytic continuation to other region of $r_l$'. We note that the integral formula for the GLSM partition function can be used as a tool for this analytic continuation.