Rate of cluster decomposition via Fermat point

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Abstract

In quantum field theory with a mass gap correlation function between two spatially separated operators decays exponentially with the distance. We argue that in a general quantum field theory the optimal suppression of a three-point function is determined by total distance from the operator locations to the Fermat point. Similarly, for the higher point functions we conjecture the optimal exponent is determined by the solution of the Euclidean Steiner tree problem.

Set up of the problem

Cluster decomposition of vacuum correlation functions is one of the basic results of quantum field theory, which underlines the locality of interactions. When a theory has a mass gap m, connected correlations between spatially separated operators $O(x)$ decay exponentially with the distance,

$$\langle O(x)O(0)\rangle \sim e^{-m|x|}, \qquad \text{for} \quad m|x| \gg 1.$$

In a relativistic case this fundamental result readily follows from i.e. Källén-Lehmann spectral representation and can be established in a number of ways:

$$\langle O(x)O(y)\rangle = \int_{m^2}^{+\infty} \frac{d\mu^2}{2\pi}\rho(\mu^2)\int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{i}{p^2-\mu^2+i\epsilon}e^{-ip(x-y)},$$

where

$$\rho(\mu^2) = 2\pi\sum_\lambda \delta(\mu^2 - m_\lambda^2)\,|\langle 0|\,O(0)\,|\lambda\rangle\,|^2$$

is spectral density. For convenience we show all the results for scalar operators.

We consider three points x_i^μ with all three mutual distances are space-like and much larger than the inverse mass gap

$$m\ell_i \gg 1, \qquad \ell_1 = |\vec{x}_2 - \vec{x}_3|, \quad \ell_2 = |\vec{x}_3 - \vec{x}_1|, \quad \ell_3 = |\vec{x}_1 - \vec{x}_2|.$$

Without loss of generality throughout this paper we assume

$$\ell_1 \geq \ell_2 \geq \ell_3.$$

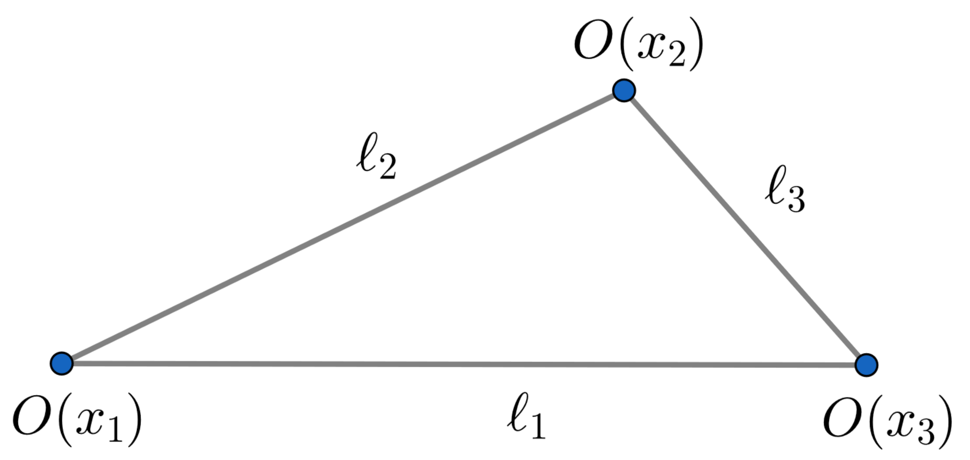

Figure 1: We are interested in suppression of three-point correlator when all the mutual distances are space-like.

We are interested in calculating exponential suppression of

$$G_{123}(x_1,x_2,x_3) = \langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle \sim e^{-m\,\ell_{\text{Fermat}}}.$$

We argue that the optimal rate of suppression, i.e. the best rate which would universally apply to all theories and operators O_i, for the three-point correlator is given by the sum of distances to the operator locations from the Fermat point.

Euclidean configuration

We start with the case when all points belong to a spatial plane, such that all three operators are simultaneous $x_i^0 = 0$.

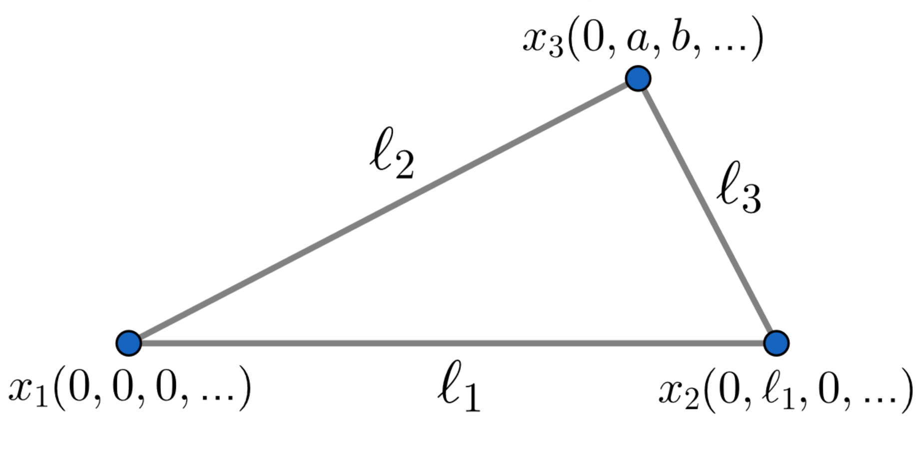

Figure 2: When the triangle inequality is satisfied, $\ell_2+\ell_3>\ell_1$, the points belong to a spatial plane and all three times can be chosen to be zero.

To establish the optimal rate of suppression we consider simplest diagrams contributing to the connected part of three-point function.

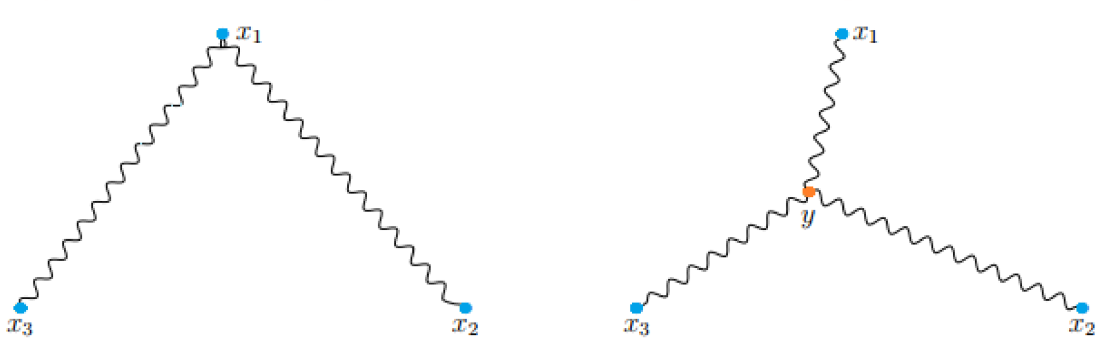

Figure 3: First class of diagrams include no additional vertexes and second class include only one.

In first class of diagrams, given that each propagator $G(x_i - x_j)$ is suppressed as $e^{-m|x_i - x_j|}$, the largest universal suppression is given by

$$G_{123}(x_1,x_2,x_3) \sim e^{-m(\ell_2+\ell_3)}.$$

This is for example the suppression rate in a theory of free massive scalar field φ when $\phi_1 = \phi, \phi_2 = \phi^2, \phi_3 = \phi$.

Another class of diagrams include one interaction vertex connected by propagators with the original operators,

$$I_{123} = \int d^d y\, G(y-x_1)G(y-x_2)G(y-x_3).$$

In principle, the propagators in I_{123} can be different, we only assume that m is the lightest excitation propagating in each channel. A crucial simplification comes from the fact that all operators are simultaneous and the integral I_{123} can be calculated in the Euclidean space.

Using Källén-Lehmann representation in the coordinate space

$$G(x) = \int_m^\infty d\mu^2\rho(\mu^2)\left(\frac{-i\mu}{4\pi^2|x|}\right)^{(d-2)/2} K_{(d-2)/2}(\mu|x|),$$

the integral of interest reduces to

$$\int d^d y\,\frac{K_{(d-2)/2}(\mu_1|y-x_1|)K_{(d-2)/2}(\mu_2|y-x_2|)K_{(d-2)/2}(\mu_3|y-x_3|)}{(|y-x_1||y-x_2||y-x_3|)^{(d-2)/2}},$$

where $\mu_i \geq m$.

The integral over d-dimensional Euclidean space can be split into the integral over three ball regions $\mu_i|y-x_i| \lesssim 1$ and the rest.

It can be shown that the integral I_{123} over the ball regions around the original operators will give the same exponential suppression factor $e^{-m(\ell_2+\ell_3)}$ as the "non-interacting" Feynman diagrams mentioned above. The divergent term is regularized by introducing an appropriate UV-cutoff.

The integral I_{123} over the rest of the d-dimensional space excluding the balls $\mu_i|y-x_i| \lesssim 1$ can be bounded by

$$I_{123} \lesssim \int d^d y\, e^{-m(|y-x_1|+|y-x_2|+|y-x_3|)},$$

This integral can be extended back to the whole Euclidean d-dimensional space, because the additional "added by hands" integrals of the exponent $e^{-m(|y-x_1|+|y-x_2|+|y-x_3|)}$ over the regions $m|y-x_i| \lesssim 1$ is suppressed by $e^{-m(\ell_k+\ell_l)}$, $i \neq k,l$ and thus unimportant. The leading exponent is given by the smallest value

$$\min_{y\in\mathbb{R}^d} |y-x_1|+|y-x_2|+|y-x_3|.$$

Clearly the minimum is achieved when y belongs to the same two-dimensional spatial plane as x_i. Hence minimization problem becomes the famous Fermat-Torricelli problem of finding a point on a plane such that total distance from the three vertexes of a given triangle to that point is the minimum possible. It is easy to see that the minimal total distance, which we denote ℓ_{Fermat} is not larger than $\ell_2+\ell_3$. Hence all terms suppressed as $e^{-m(\ell_2+\ell_3)}$ are subleading, while the optimal exponent is given by

$$I_{123} \lesssim e^{-m\,\ell_{\text{Fermat}}}$$

The expression for ℓ_{Fermat} in terms of ℓ_i will be given below.

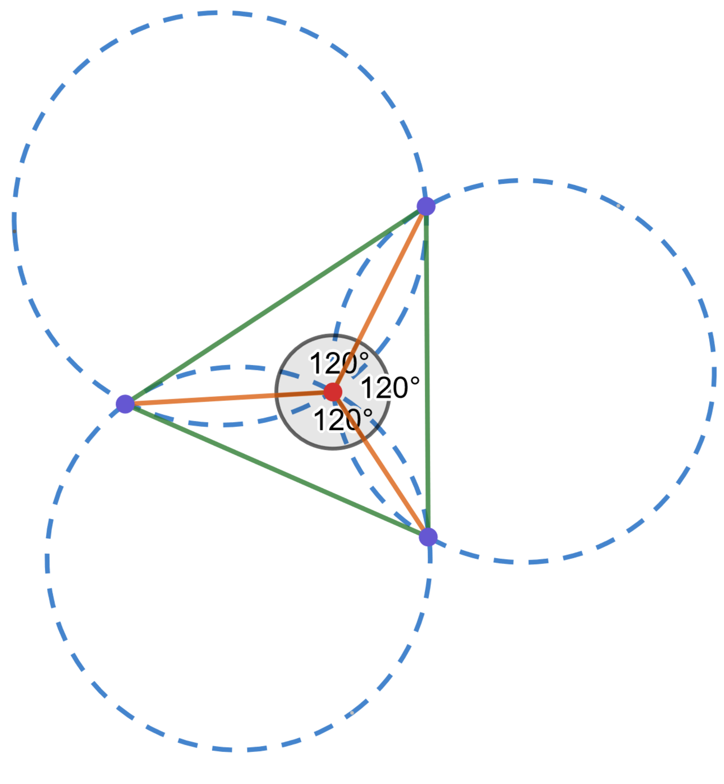

Figure 4: Fermat-Torricelli problem.

General configuration

Our goal in this section is to consider all possible configurations of three points x_i^μ in the Minkowski space with the signature $(+,-,-,\ldots)$, assuming their mutual intervals are space-like,

$$(x_2-x_3)^2 = -\ell_1^2, \quad (x_3-x_1)^2 = -\ell_2^2, \quad (x_1-x_2)^2 = -\ell_3^2$$

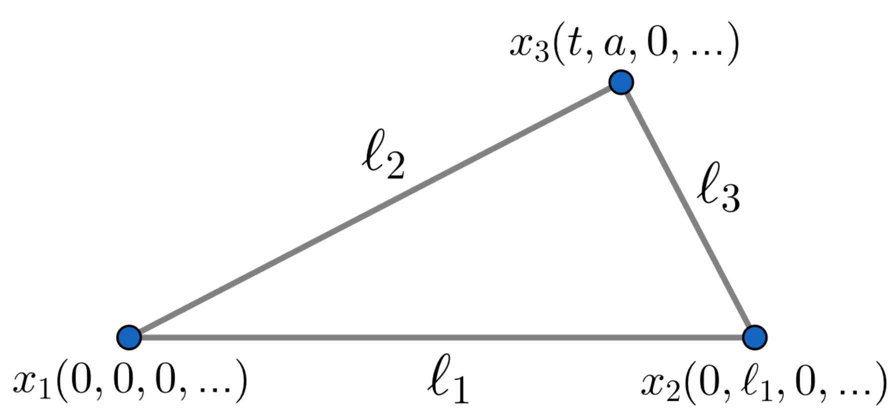

Figure 5: Simplest kinematics when the triangle inequality is violated $\ell_2+\ell_3<\ell_1$ and three points are inherently Minkowskian.

To estimate the leading exponent in case when the configuration is Minkowskian we resort to a massive φ^3 theory when all three operators are the same $O_i = \varphi$. Then the integral I_{123} written in the momentum space is given by

$$\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{e^{ik_1(x_1-x_3)+ik_2(x_2-x_3)}}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)((k_1+k_2)^2-m^2+i\epsilon)}.$$

Using Schwinger parameter representation we can reduce the integral to

$$I_{123} = \frac{i^{d-3}}{(4\pi)^d}\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\gamma\frac{1}{(\alpha\beta+\beta\gamma+\gamma\alpha)^{d/2}}$$

$$\exp\left(-\frac{i}{4}\frac{\alpha\ell_1^2+\beta\ell_2^2+\gamma\ell_3^2}{\alpha\beta+\beta\gamma+\gamma\alpha} + i(\alpha+\beta+\gamma)(-m^2+i\epsilon)\right).$$

The main contribution comes from the saddle point,

$$\ell_1^2 = 4m^2(\beta^2+\gamma^2+\beta\gamma), \ell_2^2 = 4m^2(\gamma^2+\alpha^2+\gamma\alpha), \ell_3^2 = 4m^2(\alpha^2+\beta^2+\alpha\beta),$$

provided it belongs to the octant $\alpha,\beta,\gamma \geq 0$.

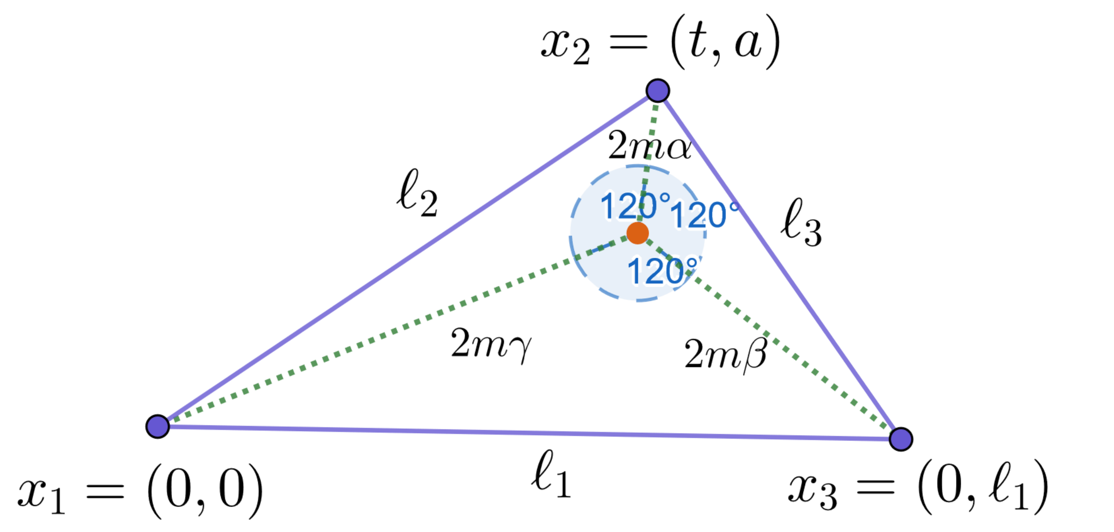

Figure 6: The sides $x_i - x_j$ of the original triangle together with the lines connecting Fermat point with the original points x_i form three triangles, each has an obtuse angle of 120°.

Introduce Feynman parameters a,b,c and Schwinger parameter t

$$a,b,c \geq 0, \qquad a+b+c = 1,$$

$$\alpha = ta, \qquad \beta = tb, \qquad \gamma = tc.$$

The integral over t can be calculated, yielding

$$I_{123} = \frac{2}{(4\pi)^d}\left(\frac{im}{l}\right)^{d-3}\int_\Delta da\,db\,\frac{K_{d-3}(ml(a,b,c))}{(ab+bc+ca)^{d/2}},$$

where

$$l^2 = \frac{a\,\ell_1^2 + b\,\ell_2^2 + c\,\ell_3^2}{ab+bc+ca}.$$

It is thus clear that the integral in the limit $m\ell_i \gg 1$ is saturated by the maximal value of $l(a,b,c)$ inside the triangle $a+b+c=1$. We start our analysis with the conventional Euclidean case when $\ell_2+\ell_3 > \ell_1$.

When all angles of the original triangle are smaller than 120°, i.e. $\ell_2^2+\ell_2\ell_3+\ell_3^2 > \ell_1^2$, the Fermat point is located strictly inside the original triangle,

$$\max_\Delta\, l(x,y,z) = \ell_{\text{Fermat}} = \sqrt{\frac{1}{2}(\ell_1^2+\ell_2^2+\ell_3^2+\sqrt{3\mathcal{D}})},$$

where $\mathcal{D} = 16S^2(\ell_1,\ell_2,\ell_3)$. Here $S(\ell_1,\ell_2,\ell_3)$ is the area of the triangle given by Heron's formula.

When $(\ell_2^2+\ell_2\ell_3+\ell_3^2)/\ell_1^2$ decreases and becomes smaller than 1, the obtuse angle becomes equal or large 120°, then the Fermat point coincides with the vertex of the obtuse angle x_1^μ. In this case the maximum of $l(a,b,c)$ is achieved at the boundary $a=0$,

$$\max_\Delta\, l(a,b,c) = \ell_{\text{Fermat}} = \ell_2+\ell_3,$$

and the contributions of both Feynman diagrams depicted in Fig.3 is of the same order.

When ℓ_1 further grows and approaches $\ell_1 = \ell_2+\ell_3$ the triangle degenerates into a spatial line or belongs to a plane with one direction being light-like. When $\ell_1 > \ell_2+\ell_3$ the triangle inequality is violated and the triangle is inherently Minkowskian, see Fig. 5. In all cases $\ell_1 \geq \ell_2+\ell_3$ the maximum of $l(a,b,c)$ is achieved on the boundary $a=0$.

Finally we have

$$\ell_{Fermat} = \begin{cases}\sqrt{\frac{1}{2}(\ell_1^2+\ell_2^2+\ell_3^2+\sqrt{3\mathcal{D}})}, & \ell_2^2+\ell_2\ell_3+\ell_3^2 > \ell_1^2,\\ \ell_2+\ell_3, & \ell_2^2+\ell_2\ell_3+\ell_3^2 \leq \ell_1^2.\end{cases}$$

Discussion

It is an interesting question to extend our consideration to a general n-point function. The leading exponent will be given by the total length of the graph solving the Euclidean-Steiner problem.

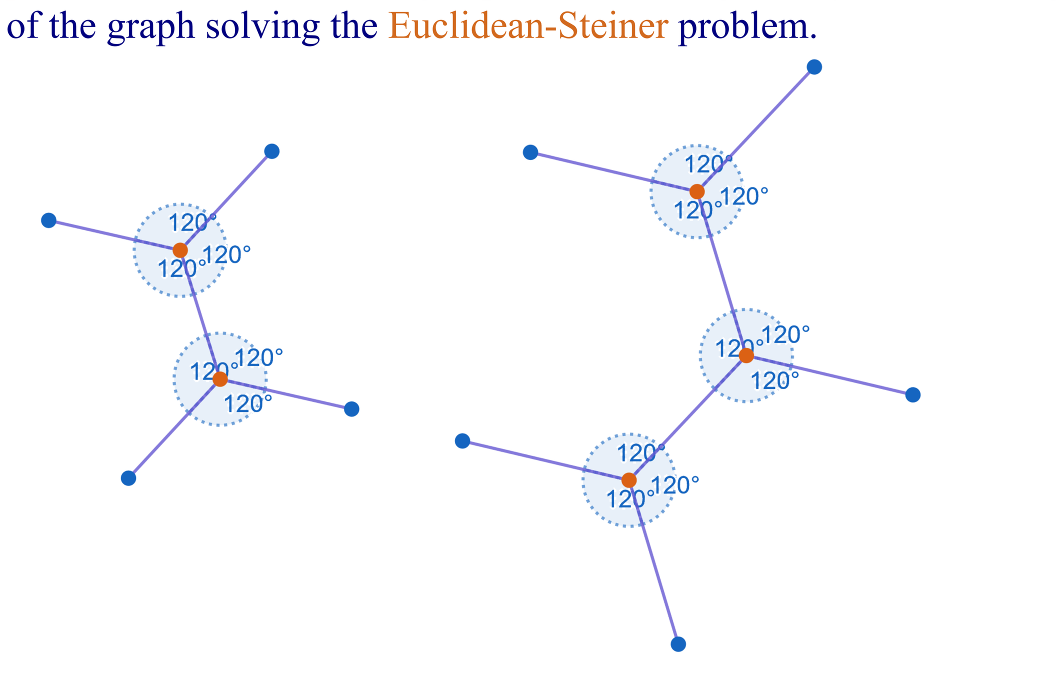

Figure 7: Euclidean-Steiner problem for four and five points graph.

References

Alexander Avdoshkin, Lev Astrakhantsev, Anatoly Dymarsky, Michael Smolkin, *"Rate of cluster decomposition via Fermat point"*, arXiv: 1811.03633.