Analytic example of the Aretakis type behavior of the metric

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Reissner-Nordstrom black holes

Reissner-Nordstrom (RN) black holes are static solutions of the Einstein-Maxwell theory and they describe a gravitational field of non-rotating black holes with zero electric charge.

Einstein-Maxwell action:

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

Equations of motion:

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0$$

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T^{\eta}{}_{\eta} \right)$$

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

Ansatz for static and spherically symmetric solutions:

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2{}_{D-2}$$

Non-zero components of field-strength tensor:

$$E = F_{tr} = -F_{rt} = f(r)$$

Plugging this into Maxwell equation one obtains:

$$\partial_r (r^{D-2} f(r)) = 0 \qquad \Rightarrow \qquad E = f(r) = -\frac{Q}{\Omega_{D-2} r^{D-2}}$$

In case D=4 from the last expression we see that Q can be interpreted as an electric charge of a black hole.

Reissner-Nordtrom solution:

$$ds^2 = -(1-\frac{2M}{r}+\frac{Q^2}{r^2})dt^2 + \frac{1}{(1-\frac{2M}{r}+\frac{Q^2}{r^2})}dr^2 + r^2 d\Omega^2$$

Two horizons coincide in extremal case when M=Q

$$ds^2 = -(1-\frac{M}{r})^2 dt^2 + \frac{1}{(1-\frac{M}{r})^2} dr^2 + r^2 d\Omega^2$$

It is convenient to make substitution $v = t + r^*$

,where $r^* = \frac{M^2}{M-r} + 2M\ln(r-M) + r$

Then RN solution rewrites in the following form:

$$ds^2 = -(1 - \frac{M}{r})^2 dv^2 + 2dvdr + r^2 d\Omega^2$$

To work with a geometry near the horizon, we need to make substitutions:

$$v \mapsto \frac{v}{\lambda} \qquad r \mapsto M + \lambda r$$

And after that take the limit $\lambda \to 0$

$$ds^2 = -\frac{r^2}{M^2} dv^2 + 2dvdr + M^2 d\Omega^2$$

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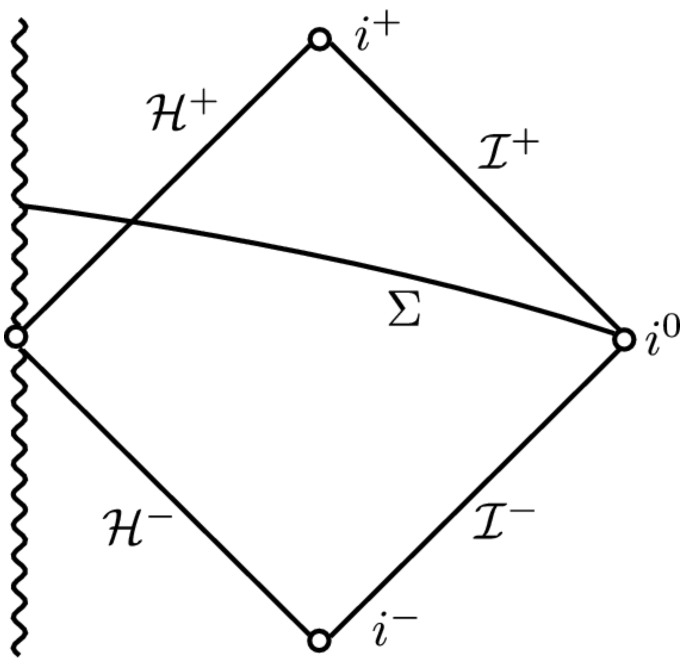
Aretakis instability

Aretakis proved that massless scalar field is unstable on the horizon of extreme RN [arXiv:1110.2007]

Equation of motion for a massless scalar field:

$$\nabla^2\psi = 0$$



We look for a solution of this equation with initial data specified on a Cauchy surface Σ intersecting H^+ and extending to infinity.

For ERN metric, after spherical modes decomposition, equation of motion reads:

$$2r\partial_v\partial_r(r\psi_l)+\partial_r((r-M)^2\partial_r\psi_l)-l(l+1)=0$$

$$l=0$$
$$r=M$$

$$\Rightarrow \quad H_0[\psi]=\frac{1}{M}[\partial_r(r\psi_0)]_{r=M}$$

- constant (w.r.t. time evolution)

$(\partial_r\psi_0)_{r=M}\to H_0$ as $v\to\infty$

It can be shown that

$$(\partial_r{}^k \psi_0)_{r=M} \propto v^{k-1} \quad \text{for large} \quad v$$

In case $l \neq 0$ constants, that are conserved on the horizon :

$$H_l[\psi] \equiv \frac{1}{M}\{\partial_r{}^l[r\partial_r(r\psi_l)]\}_{r=M}$$

$\partial_r{}^k \psi_l$ decays on and outside the horizon as $k \leq l$

$\partial_r{}^{l+1} \psi_l$ does not decay at the horizon

$\partial_r{}^{l+2} \psi_l$ grows linearly at the horizon

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Massive body infall

Einstein equation:

$$G(\tilde{g})^{ab} = T^{ab}$$

, where $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ - perturbed metric

After expanding in powers of h^{ab}

$$G(\tilde{g}(h)) = G(g) + \delta G(g,h) + \Delta G(g,h)$$

Einstein equation reads: $\delta G(g,h)^{ab} = 8\pi T^{ab}{}_{eff}$

In the last equation it's convenient to make substitution:

$$\gamma_{ab} = h_{ab} - \frac{1}{2} g_{ab} g_{cd} h^{cd}$$

Finally, equation takes the form:

$$\Box\gamma_{ab} + 2R^{c\ \ d}_{\ a\ \ b}\gamma_{cd} = -16\pi T_{ab}$$

Let us make transformation:

$$\gamma_{ij}(v,r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega \sum_{l,n} Y_{l,n}(\theta,\varphi)e^{-i\omega[v-V(r)]}\gamma_{ij,\ln}(\omega,r)$$

$$r^2\partial^2_{rr}\tilde{\gamma}_{00;\ln} - 2M^2 i\omega\partial_r\tilde{\gamma}_{00;\ln} - 2r\partial_r\tilde{\gamma}_{00;\ln} - l(l+1)\tilde{\gamma}_{00;\ln} = 4ri\omega\tilde{\gamma}_{01;\ln} - 16M^2\pi\tilde{T}_{00;\ln}$$

$$r^2\partial^2_{rr}\tilde{\gamma}_{01;\ln} - 2M^2 i\omega\partial_r\tilde{\gamma}_{01;\ln} + 2r\partial_r\tilde{\gamma}_{01;\ln} - (2 + l(l+1))\tilde{\gamma}_{01;\ln} = \frac{2r}{M^2}i\omega\tilde{\gamma}_{11;\ln} - 16M^2\pi\tilde{T}_{01;\ln}$$

$$r^2\partial^2_{rr}\tilde{\gamma}_{11;\ln} - 2M^2 i\omega\partial_r\tilde{\gamma}_{11;\ln} + 6r\partial_r\tilde{\gamma}_{11;\ln} - (4 - l(l+1))\tilde{\gamma}_{11;\ln} = -16M^2\pi\tilde{T}_{11;\ln}$$

In the near-horizon limit: $r \to 0$

$$[r^2 \frac{\partial^2}{\partial r^2} - 2i\omega M^2 \frac{\partial}{\partial r} + (2\omega^2 M^2 V'(0) - b_{ij})]\tilde{\gamma}_{ij,\ln}(\omega, r) \approx \tilde{T}_{ij,\ln}(0)$$

, where

$$b_{00} = l(l+1) \qquad b_{01} = l(l+1) + 2 \qquad b_{11} = l(l+1) - 4$$

$$\tilde{T}_{00,\ln}(0) = -8\sqrt{2\pi} m Y^*_{l,n}(\theta_0, \varphi_0)$$

$$\tilde{T}_{01,\ln}(0) = 4\frac{\sqrt{2\pi} m}{E} Y^*_{l,n}(\theta_0, \varphi_0)$$

$$\tilde{T}_{11,\ln}(0) = -\frac{2\sqrt{2\pi} m}{E^3} Y^*_{l,n}(\theta_0, \varphi_0)$$

Particular solutions of this differential equations are:

$$\tilde{\gamma}_{00,\ln}(\omega,0) = -\frac{\tilde{T}_{00,\ln}(0)}{\omega^2 M^2 E^{-2} + l(l+1)}$$

$$\tilde{\gamma}_{01,\ln}(\omega,0) = -\frac{\tilde{T}_{01,\ln}(0)}{\omega^2 M^2 E^{-2} + l(l+1) + 2}$$

$$\tilde{\gamma}_{11,\ln}(\omega,0) = -\frac{\tilde{T}_{11,\ln}(0)}{\omega^2 M^2 E^{-2} + l(l+1) - 4}$$

Performing inverse Fourier tranform, one obtains:

$$\gamma_{00;\ln}(v,0) = -\frac{\widetilde{T}_{00;\ln}(0)E}{M}\sqrt{\frac{\pi}{2l(l+1)}}\; e^{-\frac{E}{M}\sqrt{l(l+1)}(v-C_0)}\theta(v-C_0)$$

$$\gamma_{01;\ln}(v,0) = -\frac{\widetilde{T}_{01;\ln}(0)E}{M}\sqrt{\frac{\pi}{2[l(l+1)+2]}}\; e^{-\frac{E}{M}\sqrt{l(l+1)+2}(v-C_0)}\theta(v-C_0)$$

$l \neq 0,1$

$$\gamma_{11;\ln}(v,0) = -\frac{\widetilde{T}_{11;\ln}(0)E}{M}\sqrt{\frac{\pi}{2[l(l+1)-4]}}\; e^{-\frac{E}{M}\sqrt{l(l+1)-4}(v-C_0)}\theta(v-C_0)$$

$l = 0,1$

$$\gamma_{11;\ln}(v,0) = \frac{\widetilde{T}_{11;lm}(0)E}{M}\sqrt{\frac{2\pi}{4-l(l+1)}}\sin[\frac{E}{M}\sqrt{4-l(l+1)}(v-C_0)]\theta(v-C_0)$$

Thank you for attention!