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Quantum field theory in an external scalar field

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Introduction

- We consider the Yukawa model of interacting massive Dirac field and massless scalar field in $(1+1)$-dimensional Minkowski space-time with $(1,-1)$ signature:

$$S = \int dtdx \left[\frac{1}{2}(\partial_\mu \phi)^2 + i\bar{\psi}\partial\!\!\!/\psi - \lambda\phi\bar{\psi}\psi \right], \tag{1}$$

- We use the following representation of the Clifford algebra:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}_{2\times 2}. \tag{2}$$

- Varying the action (1) over ϕ and ψ one obtains the equations of motion:

$$\begin{cases} \partial_\mu \partial^\mu \phi = -\lambda\bar{\psi}\psi, \\ i\partial\!\!\!/\psi = \lambda\phi\psi, \end{cases} \tag{3}$$

  with classical solution $\phi_{cl} = \alpha + \beta x, \ \ \psi_{cl} = 0$, where we denote $b = \lambda\beta$ and $a = \lambda\alpha$.
- We use the classical solution $\phi_{cl} = \alpha + \beta x$ as an external background for the Dirac field.

Mode solutions

- For the function $\psi(x,\omega)$ one can get the equations in components:

$$\begin{cases} \left[\partial_x^2 - (a+bx)^2 + \omega^2 - b\right]\psi_1(x,\omega) = 0 \\ \left[\partial_x^2 - (a+bx)^2 + \omega^2 + b\right]\psi_2(x,\omega) = 0 \end{cases} \tag{4}$$

- Exact solutions of this equations is the sum of parabolic cylinder functions $D(\nu, z)$, [1, 2, 3]:

$$\psi_1 = \frac{1}{2}\left\{ e^{\frac{i\pi\omega^2}{4b}} e^{-\frac{\omega^2}{4b} + \frac{\omega^2}{4b}\log\frac{\omega^2}{2b}} D\left(-\frac{\omega^2}{2b}, i\sqrt{\frac{2}{b}}(a+bx)\right) - \frac{i|\omega|}{\sqrt{2b}} e^{\frac{\omega^2}{4b} - \frac{\omega^2}{4b}\log\frac{\omega^2}{2b}} D\left(-1+\frac{\omega^2}{2b}, \sqrt{\frac{2}{b}}(a+bx)\right)\right\} \tag{5}$$

$$\psi_2 = \frac{i}{2}\left\{ e^{\frac{i\pi\omega^2}{4b}} e^{-\frac{\omega^2}{4b} + \frac{\omega^2}{4b}\log\frac{\omega^2}{2b}} \frac{|\omega|}{\sqrt{2b}} D\left(-1-\frac{\omega^2}{2b}, i\sqrt{\frac{2}{b}}(a+bx)\right) - e^{\frac{\omega^2}{4b} - \frac{\omega^2}{4b}\log\frac{\omega^2}{2b}} D\left(\frac{\omega^2}{2b}, \sqrt{\frac{2}{b}}(a+bx)\right)\right\} \tag{6}$$

- Here we have chosen integration constants knowing anticommutation relations

$$\{\hat{\psi}(t,x), \hat{\psi}^\dagger(t,y)\} = \delta(x-y)\cdot \mathbf{1}_{2\times 2} \tag{7}$$

and wanting the modes to be plane in the limit of $\omega \to \infty$. Using the asymptotic normalization method we get

$$\psi_1(x,\omega) = \frac{e^{i|\omega|x}}{\sqrt{2}} e^{i\varphi(\omega)} \quad \text{for} \quad \omega \to \infty, \quad \text{where } \varphi(\omega) \text{ is an arbitrary phase.} \tag{8}$$

- (4) $\Rightarrow$ the negative-frequency modes are obtained from the positive-frequency ones by complex conjugation.

Tree-level current

- Calculate the normal-ordered current in the limit of $x \to \infty$:

$$\langle :\bar{\psi}\psi:\rangle = \langle\bar{\psi}\psi\rangle - \langle\bar{\psi}\psi\rangle_{\beta=0}\,. \tag{9}$$

Here we represent

$$\hat{\psi}(x,t) = \int (d\omega) e^{-i\omega t}\left[\hat{b}_\omega u_\omega(x) + \hat{c}^\dagger_{-\omega} v_{-\omega}(x)\right],\ \text{where } u_\omega(x) = \begin{pmatrix}\psi_1(x,\omega)\\ \psi_2(x,\omega)\end{pmatrix},\ v_\omega(x) = \begin{pmatrix}\widetilde{\psi}_1(x,\omega)\\ \widetilde{\psi}_2(x,\omega)\end{pmatrix}, \tag{10}$$

- Without an external field, one can get an exact expression for the modes and get such a "free"current:

$$\langle\bar{\psi}\psi\rangle_{\beta=0} = \int (d\omega)\frac{m}{\sqrt{\omega^2 - m^2}} \tag{11}$$

- Introduce an UV cut-off Λ, which is much larger than any frequency in the system, and even $\Lambda \gg M(x)$, where we denoted $M(x) = (a + bx)$ for short.

$$\langle :\bar{\psi}\psi:\rangle = 2\int_m^\Lambda (d\omega)\left[\widetilde{\psi}_1\widetilde{\psi}_2^* + \widetilde{\psi}_1^*\widetilde{\psi}_2 - \frac{m}{\sqrt{\omega^2 - m^2}}\right] \approx -\frac{bx}{\pi}\log\frac{bx}{\Lambda} \tag{12}$$

In order to calculate it we have used the asymptotics of parabolic cylinder functions for large values of the parameter and for large values of the argument, [1, 2, 3].

Effective action

- Calculate the effective action and get the current from it. For convenience, we make the Wick rotation into the Euclidean space. Making some conversions in the functional integral we get

$$Z_{\mathsf{E}}[\phi] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{-\int d^2x \bar{\psi}\left(i\partial\!\!\!/\psi - \lambda\phi\right)\psi\right\} = e^{-S_{eff}[\phi]}, \tag{13}$$

where

$$S_{eff}[\phi] = -\mathrm{tr}\log\frac{i\partial\!\!\!/ - \lambda\phi}{i\partial\!\!\!/} \tag{14}$$

- Introduce an ultraviolet cut-off at the scale $\Lambda \gg m$ and calculate (14) in the limit $\Lambda \gg \lambda\phi_{cl}$:

$$S_{eff}[\phi] = -\int d^2x\left[\frac{(\lambda\phi)^2}{2\pi}\log\frac{\Lambda}{\lambda\phi} + \frac{(\lambda\phi)^2}{4\pi} - \frac{m^2}{2\pi}\log\frac{\lambda\phi}{m}\right] \tag{15}$$

From here one can get the tree-level current

$$\langle\bar{\psi}\psi\rangle = \frac{1}{\lambda}\frac{\delta}{\delta\phi}e^{-S_{eff}[\phi]}\Big|_{\phi_{\mathsf{cl}}} \approx \frac{1}{\lambda}\frac{\lambda^2\phi_{\mathsf{cl}}}{\pi}\log\frac{\Lambda}{\lambda\phi_{\mathsf{cl}}} = -\frac{\lambda\phi_{\mathsf{cl}}}{\pi}\log\frac{\lambda\phi_{\mathsf{cl}}}{\Lambda} \tag{16}$$

Conclusion

- We have calculated the current $\langle \bar{\psi}\psi \rangle$ in two ways and the results, (12) and (16), coincide.
- This result for the current shows that the field ϕ should drop down to the minimum of the effective potential due to

$$\Box \langle \phi \rangle = -\lambda \left\langle \bar{\psi}\psi \right\rangle . \tag{17}$$

  The obtained effective Lagrangian is a two-dimensional scalar analogue of the Heisenberg-Euler Lagrangian.
- The next step is to make the field ϕ dynamical and to calculate loop corrections using Schwinger-Keldysh diagrammatic technique.

References

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