Imaginary parts of Gaussian effective actions in de Sitter space

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Motivation

We denote the mass of massive real scalar field $\varphi$ as $m$. Because of the relation:

$$\langle \text{Out} | \text{In} \rangle = e^{i \int L_{\text{eff}} dx}, \quad \text{and} \quad L_{\text{eff}} = \int_{\infty}^{m^2} dm^2 \ G_F(x, x),$$

if $L_{\text{eff}}$ is real this transition amplitude will be some phase and the probability of the transition from the In- to the Out- state is equal to one. But if the effective Lagrangian has an imaginary part the probability of such a transition is not equal to one:

$$\left| \langle \text{Out} | \text{In} \rangle \right|^2 \neq 1,$$

which usually signals a particle creation!.
Space-time, metric and equation of motion

Consider D-dimensional global de Sitter space \( (R = 1) \) with the following metric:

\[
ds^2 = -dt^2 + \cosh^2(t) d\Omega^2.
\]

Klein-Gordon equation:

\[
\left(\partial_t^2 + (D - 1) \tanh(t) \partial_t + j(j + D - 2) \cosh^{-2}(t) + m^2\right) \varphi_j(t) = 0.
\]

This equation has two linear independent solutions:

\[
\varphi_j(t) = \alpha_1 \text{ch}(t) - \frac{D-1}{2} \text{P}^{-i\mu}_{j + \frac{D-3}{2}}(\tanh t) + \alpha_2 \frac{2}{\pi} \text{ch}(t) - \frac{D-1}{2} \text{Q}^{-i\mu}_{j + \frac{D-3}{2}}(\tanh t).
\]

Spherical harmonics expansion is performed

\[
\varphi = \sum_{j,m} \varphi_j(t) Y_{jm}(\Omega), \quad \mu^2 = m^2 - \frac{(D - 1)^2}{4}.
\]

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Here and below $\vec{x}$ is a vector of angular coordinates on $(D - 1)$-dimensional sphere. Consider the field operator ($\tilde{t} \equiv \tanh t$):

$$\hat{\varphi}(t, \vec{x}) = \sum_{j, m} \text{ch}(t)^{-\frac{D-1}{2}} \left[ \left( \alpha_1 P_\nu^{-i\mu}(\tilde{t}) + \alpha_2 \frac{2}{\pi} Q_\nu^{-i\mu}(\tilde{t}) \right) Y_{jm}(\vec{x}) \hat{a}_{j, m} + \text{h.c.} \right].$$

Annihilation and creation operators:

$$[\hat{a}_{j, m}, \hat{a}^\dagger_{j', m'}] = \delta_{j, j'} \delta_{m, m'}.$$

Canonical commutation relations:

$$[\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{y})] = i \frac{\delta(\vec{x} - \vec{y})}{\sqrt{g}} \quad \rightarrow \quad \alpha_1^2 + \alpha_2^2 = \frac{\pi}{2 \sinh(\mu \pi)}$$

This is the condition, which should be obeyed by $\alpha_{1,2}$ coefficients to have the canonical commutation relations.
Behavior of modes at plus and minus infinity

One can find asymptotic expansion for modes. For example:

\[ P^{-i\mu}(\tanh t) \approx C_+ e^{i\mu t} + C_- e^{-i\mu t}, \quad \text{as} \quad t \to -\infty. \]

So, modes behave like waves at \( t \to \pm \infty \). We are interested in single wave behavior. Hence one wave at plus infinity (\textbf{Out-modes}) corresponds to:

\[ \alpha_1 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \text{and} \quad \alpha_2 = 0. \]

At the same time the one wave at minus infinity (\textbf{In-modes}) corresponds to:

\[ \alpha_1 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \alpha_2 = 0, \quad \text{in odd dimensions,} \]

and \( \alpha_2 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \alpha_1 = 0, \quad \text{in even dimensions.} \)
Out- mode even

In- mode even
In- and Out- mode odd
Vacuum states

Vacuum state is defined as:

\[ a_{j,m} |\alpha\rangle = 0. \]

It means that:

different \( \alpha_1, \alpha_2 \) → different mode expansion → different creation and annihilation operators → different ground states.

We consider two states:

\[ |\text{In}\rangle \quad \text{singe wave at past infinity} \]

\[ |\text{Out}\rangle \quad \text{single wave at future infinity} \]
Feynman In-Out- propagator in even dimensions

\[ G_{\text{In-Out}}(t_1, \vec{x} | t_2, \vec{y})^{\text{even}} = \frac{\langle \text{Out} | T \hat{\phi}(\vec{y}, t_2) \hat{\phi}(\vec{x}, t_1) | \text{In} \rangle}{\langle \text{Out} | \text{In} \rangle} = \]

\[ = - \frac{i (-1)^{\frac{D-2}{2}}}{2 (2\pi)^{\frac{D}{2}} \cosh \mu \pi} \left[ (Z_+^2 - 1)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}} - i \mu - \frac{1}{2} \left( -Z_+ \right) + (Z_-^2 - 1)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}} - i \mu - \frac{1}{2} \left( -Z_- \right) \right] \]

\[ \text{Im } G_{\text{In-Out}}^{\text{even}} (Z = 1) = - \frac{(-1)^{\frac{D-2}{2}} |\Gamma\left(\frac{D-1}{2} + i \mu \right)|^2}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right) \cosh \pi \mu} \]

\[ Z_\pm \equiv Z \pm i \epsilon = \frac{-\tanh t_1 \tanh t_2 + \vec{x} \vec{y}}{\sqrt{1 - (\tanh t_1)^2} \sqrt{1 - (\tanh t_2)^2}} \pm i \epsilon. \]
Another interpretation

One can convert effective action into the quantum mechanical path integral:

\[ iS_{\text{eff}} = \log \left( \int d[\varphi] e^{i \int d^d x \mathcal{L}} \right) = \int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)} d[x] e^{i \int_0^T dt \left( \frac{\dot{x}^2}{4} + m^2 \right)} = \]

\[ = \int_0^\infty \frac{dT}{T} e^{i S_{\text{extremal}}} \sqrt{\frac{(2\pi i)^d}{\det (\triangle_1)}}, \]

Usually one calculates such an integral via the Wick rotating from de Sitter to Euclidean sphere, one obtain that geodesic on sphere is equator.
Another interpretation

On the D-sphere exist (D-1) direction to shrink the geodesic, that corresponds to the fact that there are (D-1) negative eigenvalue. So:

\[-S_{\text{eff}}^E \sim \sqrt{\det(\triangle_1)} \sim (-1)^{\frac{d-1}{2}}.\]

Consequently for \textbf{even} dimension \(\text{Im}\left([S_{\text{eff}}) \neq 0]\right)\) and \textbf{vanish for odd}. 

geodesic on sphere
Thank you for your attention!