Imaginary parts of Gaussian effective actions in de Sitter space

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Motivation

We denote the mass of **massive real scalar field** φ as m. Because of the relation:

$$\left\langle \text{Out} \middle| \text{In} \right\rangle = e^{i\int \mathcal{L}_{eff} dx}, \text{ and } \qquad \mathcal{L}_{eff} = \int_{\infty}^{m^2} dm^2 \; G_F(x,x),$$

if $\mathcal{L}_{eff}$ is real this transition amplitude will be some phase and the probability of the transition from the In- to the Out- state is equal to one. But if the effective Lagrangian has an imaginary part the probability of such a transition is not equal to one:

$$\boxed{\left|\left\langle \text{Out} \middle| \text{In} \right\rangle\right|^2 \neq 1},$$

which usually signals a **particle creation!**.

Space-time, metric and equation of motion

Consider D-dimensional global de Sitter space ($R = 1$) with the following metric:

$$ds^2 = -dt^2 + \cosh^2(t) d\Omega^2.$$

Klein-Gordon equation:

$$\left(\partial_t^2 + (D-1)\tanh(t)\partial_t + j(j+D-2)\cosh^{-2}(t) + m^2\right)\varphi_j(t) = 0.$$

This equation has **two linear independent solutions**:

$$\varphi_j(t) = \alpha_1 \mathrm{ch}(t)^{-\frac{D-1}{2}} \mathrm{P}^{-i\mu}_{j+\frac{D-3}{2}}(\tanh t) + \alpha_2 \frac{2}{\pi} \mathrm{ch}(t)^{-\frac{D-1}{2}} \mathrm{Q}^{-i\mu}_{j+\frac{D-3}{2}}(\tanh t).$$

Spherical harmonics expansion is performed

$$\varphi = \sum_{j,m} \varphi_j(t) Y_{jm}(\Omega), \qquad \mu^2 = m^2 - \frac{(D-1)^2}{4}.$$

Field operator

Here and below $\vec{x}$ is a vector of angular coordinates on $(D-1)$ - dimensional sphere. Consider the field operator ($\tilde{t} \equiv \tanh t$):

$$\hat{\varphi}(t,\vec{x}) = \sum_{j,m} \mathrm{ch}(t)^{-\frac{D-1}{2}} \left[\left(\alpha_1 \mathrm{P}_\nu^{-i\mu}(\tilde{t}) + \alpha_2 \frac{2}{\pi} \mathrm{Q}_\nu^{-i\mu}(\tilde{t}) \right) Y_{jm}(\vec{x}) \hat{a}^\dagger_{j,m} + h.c. \right].$$

Annihilation and creation operators:

$$[\hat{a}_{j,m}, \hat{a}^\dagger_{j',m'}] = \delta_{j,j}\delta_{m,m'}.$$

Canonical commutation relations:

$$[\varphi(t,\vec{x}), \dot{\varphi}(t,\vec{y})] = i\frac{\delta(\vec{x}-\vec{y})}{\sqrt{g}} \qquad \rightarrow \qquad \alpha_1^2 + \alpha_2^2 = \frac{\pi}{2\sinh(\mu\pi)}$$

This is the **condition**, which should be obeyed by $\alpha_{1,2}$ coefficients to have the canonical commutation relations.

Behavior of modes at plus and minus infinity

One can find asymptotic expansion for modes. For example:

$$\mathrm{P}_{\nu}^{-i\mu}(\tanh t) \approx C_{+}e^{i\mu t} + C_{-}e^{-i\mu t}., \quad \text{as} \;\; t \to -\infty.$$

So, modes behave like waves at $t \to \pm\infty$. We are interested in single wave behavior. Hence one wave at plus infinity (**Out-modes**) corresponds to:

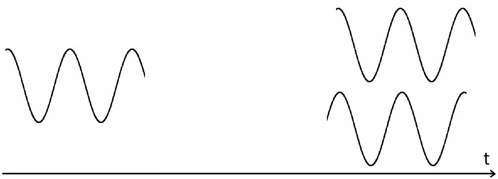
$$\alpha_1 = \sqrt{\frac{\pi}{2\sinh(\mu\pi)}}, \;\; \text{and} \;\; \alpha_2 = 0.$$

At the same time the one wave at minus infinity (**In-modes**) corresponds to:

$$\alpha_1 = \sqrt{\frac{\pi}{2\sinh(\mu\pi)}}, \qquad \alpha_2 = 0, \qquad \text{in odd dimensions},$$

$$\text{and} \quad \alpha_2 = \sqrt{\frac{\pi}{2\sinh(\mu\pi)}}, \qquad \alpha_1 = 0, \qquad \text{in even dimensions}.$$

Out- mode **even**



In- mode **even**

In- and Out- mode **odd**

Vacuum states

Vacuum state is defined as:

$$a_{j,m}\Big|\alpha\Big\rangle = 0.$$

It means that:
different $\alpha_{1,2} \to$ different mode expansion $\to$
different creation and annihilation operators $\to$ different ground states.
We consider two states:

$\Big|\mathrm{In}\Big\rangle$ singe wave at past infinity

$\Big|\mathrm{Out}\Big\rangle$ single wave at future infinity

Feynman In-Out- propagator in even dimensions

$$G_{\text{In-Out}}(t_1, \vec{x}|t_2, \vec{y})^{\text{even}} = \frac{\left\langle \text{Out}\middle| T\hat{\varphi}(\vec{y}, t_2)\hat{\varphi}(\vec{x}, t_1)\middle|\text{In}\right\rangle}{\left\langle \text{Out}\middle|\text{In}\right\rangle} =$$

$$= -\frac{i(-1)^{\frac{D-2}{2}}}{2(2\pi)^{\frac{D}{2}}\text{ch}\mu\pi}\left[\left(Z_+^2-1\right)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}}_{-i\mu-\frac{1}{2}}\left(-Z_+\right)+\left(Z_-^2-1\right)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}}_{-i\mu-\frac{1}{2}}\left(-Z_-\right)\right]$$

$$\boxed{\text{Im } G^{\text{even}}_{\text{In-Out}}(Z=1) = -\frac{(-1)^{\frac{D-2}{2}}\left|\Gamma\left(\frac{D-1}{2}+i\mu\right)\right|^2}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)\cosh\pi\mu}.}$$

$$Z_\pm \equiv Z \pm i\epsilon = \frac{-\tanh t_1 \tanh t_2 + \vec{x}\vec{y}}{\sqrt{1-(\tanh t_1)^2}\sqrt{1-(\tanh t_2)^2}} \pm i\epsilon.$$

Another interpretation

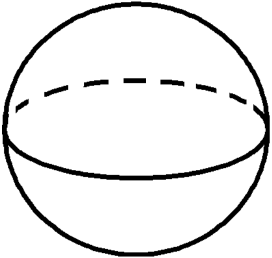
One can convert effective action into the quantum mechanical path integral:

$$iS_{eff} = \log\left(\int d[\varphi] e^{i\int d^d x \mathcal{L}}\right) = \int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)} d[x] e^{i\int_0^T dt\left(\frac{\dot{x}^2}{4}+m^2\right)} =$$

$$= \int_0^\infty \frac{dT}{T} e^{iS_{\text{extremal}}} \sqrt{\frac{(2\pi i)^d}{\det\left(\triangle_1\right)}},$$

Usually one calculates such an integral via the Wick rotating from de Sitter to Euclidean sphere, one obtain that geodesic on sphere is equator.

Another interpretation



geodesic on sphere

On the D-sphere exist (D-1) direction to shrink the geodesic, that corresponds to the fact that there are (D-1) negative eigenvalue. So:

$$-S^E_{eff} \sim \sqrt{\det\left(\triangle_1\right)} \sim (-1)^{\frac{d-1}{2}}.$$

Consequently for **even** dimension Im $\left[\left(S_{eff}\right) \neq 0\right]$ and **vanish for odd**.

Thank you for your attention!