Quantum decoherence during inflation

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1. Motivation

- During inflation the universe rapidly expands and the observed classical distribution of inhomogeneities originates from the substantially non-classical state.
- The problem of transition from quantum to classical behavior can be solved in the context of the theory of decoherence induced by environment.
- We are taking into account that the long wavelength perturbations become unobservable and considering decoherence of background degrees of freedom while maintaining information about short wavelength perturbations.

2. Inflaton field with quantum fluctuations

The action of the scalar field, minimally coupled to gravity, has the form:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\varkappa}R + \frac{1}{2}g^{\mu\nu}\partial_\nu\Phi\partial_\mu\Phi - V(\overline{\Phi})\right)$$

Let us consider inhomogenious perturbations over a flat FLRW metric and homogeneous scalar field background:

$$ds^2 = e^{2\alpha(\eta)}\{(2A-1)d\eta^2 + 2(\partial_i B)dx^i d\eta + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + h_{ij}]dx^i dx^j\}$$
$$\Phi(\vec{x},t) = \overline{\Phi}(t) + \phi(\vec{x},t)$$

Where A, B, ψ, E — scalar functions, $h_{ij} = h_{ji}$, $h^i_i = 0$ and $\partial^i h_{ij} = 0$
After substituting perturbed metric and scalar field to the Hamiltonian, it decomposes into individual components:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_S + \mathcal{H}_T,$$

Introduce the following notation:

$$\Phi_B(\vec{x},\eta) = A + e^{-\alpha(\eta)}\left[e^{\alpha(\eta)}(B-E')\right]' \text{ – Bardeen potential}$$
$$\phi^{(gi)}(\vec{x},\eta) = \phi + \Phi'(B-E') \text{ – Gauge-invariant scalar}$$
$$\left.\begin{aligned} v_S(\vec{x},\eta) &= e^{\alpha}\left[\phi^{(gi)+}\Phi'\frac{\Phi_B}{\mathscr{H}}\right] \\ v^{(\lambda)}_{T,k}(\vec{x},\eta) &= \frac{e^{\alpha}h^{(\lambda)}_k}{\sqrt{2\varkappa}}\end{aligned}\right\} \text{—Mukhanov–Sasaki variable}$$
$$z = \frac{\partial\overline{\Phi}}{\partial\eta}\frac{\partial\eta}{\partial\alpha}e^{\alpha}$$
$$\omega^2_{S,k} = k^2 - \frac{z''}{z}$$
$$\omega^2_{T,k} = k^2 - \frac{a''}{a} = k^2 - (\alpha')^2 - \alpha''$$

then the Hamiltonians take the form:

$$\hat{\mathcal{H}}_0 = e^{-2\alpha}\left[-\frac{\varkappa}{12\mathscr{L}^3}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2\mathscr{L}^3}\frac{\partial^2}{\partial\overline{\Phi}^2} + \mathscr{L}^3 V(\overline{\Phi})e^{6\alpha}\right]$$
$$\hat{\mathcal{H}}_S = \sum_k \hat{\mathcal{H}}_{S,k} = \sum_k\left(-\frac{1}{2}\frac{\partial^2}{\partial v^2_{S,k}} + \frac{1}{2}v^2_{S,k}\omega^2_{S,k}\right)$$
$$\hat{\mathcal{H}}_T = \sum_\lambda\sum_k \hat{\mathcal{H}}_{T,k} = \sum_{k,\lambda}\left(-\frac{1}{2}\frac{\partial^2}{\partial\left(v^{(\lambda)}_{T,k}\right)^2} + \frac{1}{2}\left(v^{(\lambda)}_{T,k}\right)^2\omega^2_{T,k}\right)$$

Finally, master Wheeler-DeWitt equation is:

$$\left(e^{-2\alpha}\left[-\frac{\varkappa}{12\mathscr{L}^3}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2\mathscr{L}^3}\frac{\partial^2}{\partial\overline{\Phi}^2} + \mathscr{L}^3V(\overline{\Phi})e^{6\alpha}\right] + \sum_{k,S,T,\lambda}\left(-\frac{\partial^2}{\partial v_k^2} + v_k^2\omega_k^2\right)\right)\Psi_0(\alpha,\overline{\Phi}) = 0$$

Since we believe that the perturbations are small and different modes do not interact, the solution can be sought in the form of the Born-Oppenheimer approximation [1]:

$$\Psi\left[\alpha,\overline{\Phi},\left\{v_{S,\vec{k}}, v^{(\lambda)}_{T,\vec{k}}\right\}\right] \approx \Psi_0\prod_{\vec{k},\lambda}\Psi_{s,\vec{k}}\left(v_{S,\vec{k}}|\alpha,\overline{\Phi}\right)\Psi_{T,\vec{k}}\left(v^{(\lambda)}_{T,\vec{k}}|\alpha,\overline{\Phi}\right)$$

For the homogeneous part we use the semiclassical WKB approximation:

$$\Psi_0(\alpha,\overline{\Phi}) \approx \mathcal{A}(\alpha,\overline{\Phi})e^{i\mathscr{L}^3S_0(\alpha,\overline{\Phi})-\mathscr{L}^3\mathcal{R}(\alpha,\overline{\Phi})},\ |\triangledown\mathcal{R}| \ll |\triangledown S_0|$$

The $\mathcal{R}$ function provides the wave packet shape and is associated with the $\mathcal{A}$ function. Further, we assume that $|\triangledown\mathcal{R}| \ll |\triangledown S_0|$, which corresponds to the center of the wave packet.
We also introduce the conformal WKB time [2, 3]:

$$\frac{\partial}{\partial\eta} := e^{-2\alpha}\left[-\frac{\varkappa}{6}\left(\frac{\partial S_0}{\partial\alpha}\right)\frac{\partial}{\partial\alpha} + \left(\frac{\partial S_0}{\partial\overline{\Phi}}\right)\frac{\partial}{\partial\overline{\Phi}}\right]$$

It goes along classical trajectories and in the first approximation it can be identified with the classical conformal time.

In terms of the WKB time, the Wheeler-DeWitt equation for the fluctuation part of $\Psi_{\vec{k}}$ takes the form:

$$i\frac{\partial}{\partial\eta}\Psi_{\vec{k}}(v_{\vec{k}}|\alpha,\overline{\Phi}) = \left[-\frac{1}{2}\frac{\partial^2}{\partial v^2_{\vec{k}}} + \frac{1}{2}\omega^2_{\vec{k}}(\eta)v^2_{\vec{k}}\right]\Psi_{\vec{k}}(v_{\vec{k}}|\alpha,\overline{\Phi}),$$

where $\omega^2_{\vec{k}}(\eta)$ is calculated on the classical trajectory passing through $(\alpha,\overline{\Phi})$ in the direction given by S_0. We can see that it is the Schrödinger equation.

Since some of the recent measurements of the anisotropy of the relict background did not show deviations of the primary fluctuations from the Gaussian form [4], we assume that the scalar and tensor perturbations are in the ground state, which allows us to use the ansatz in the form of a Gaussian distribution to find a solution to the Schrödinger equation.

$$\Psi_{\vec{k},0} = N_{\vec{k},0}(\eta)e^{-\frac{1}{2}\Omega_{\vec{k},0}(\eta)v_k^2}$$

In order to fix the solution, we note that the first derivatives are absent in the equation, which means that, according to the Liouville-Ostrogradsky formula, the Wronsky determinant is $\mathcal{W}_f = \dot{f}^*f - f^*\dot{f} = const$. We choose the normalization $\mathcal{W}_f = i$. We also impose the asymptotic condition $f \xrightarrow[\eta\to-\infty]{} \frac{e^{-ik\eta}}{\sqrt{2k}}$.

As mentioned above, the Mukhanov-Sasaki variable obeys the Klein-Gordon-Fock equation with a time-dependent potential on the background of the Minkowski metric [5], and its wave function evolves as an oscillator with a time-dependent frequency. Then for $\hat{v}_k$ we can introduce the creation and annihilation operators. To search for the vacuum state, we use the fact that the annihilation operator, acting on it, yields zero. Then the equation for the vacuum state $\hat{a}\Psi_{\vec{k},0} = 0$ can be written as:

$$\left[f^*\frac{\partial}{\partial v_k} - i\dot{f}^*v_k\right]\Psi_{\vec{k},0}(v_k) = 0$$

It turns out that in order for the solution to satisfy this equation while maintaining the norm $\int_{-\infty}^{\infty}dv|\Psi_0(v)|^2 = 1$, it should look like:

$$\Psi_{\vec{k},0}(v_k) = \frac{1}{(2\pi)^{1/4}}\frac{1}{\sqrt{f^*}}e^{\frac{i}{2}\frac{\dot{f}^*}{f^*}v_k^2} \qquad (1)$$

3. Power-law inflation

Let us consider the power-law inflation $e^{\alpha} \sim t^p$ with slow-roll condition, i.e. $\dot{\Phi} \ll V(\Phi),\ \ddot{\Phi} \ll 3H\dot{\Phi}$. It can be parametrized by $\varepsilon = -\frac{\dot{H}}{h^2},\ \delta = -\frac{\ddot{\Phi}}{H\dot{\Phi}}$. In this case, it turns out that $\varepsilon = \delta = \frac{1}{p} = const$. And if we use conformal time, then $e^{\alpha} \sim \eta^{1+\beta}$.
Thus, frequencies in Muhanov-Sasaki equation can be rewritten in the following way

$$\omega^2_{S,k} = \omega^2_{T,k} = \vec{k}^2 - \frac{2+3\varepsilon}{\eta^2} = \vec{k}^2 - \frac{\beta(1+\beta)}{\eta^2}, \qquad \text{where } (1+\beta) = \frac{p}{1+p}$$

Then the equation for classical fluctuations is

$$\ddot{f}_{\vec{k}} + \left(\vec{k}^2 - \frac{\beta(1-\beta)}{\eta^2}\right)f_{\vec{k}} = 0$$

This is Bessel equation, and its solution can be written in terms of the Hankel functions as:

$$f_{\vec{k}} = C_1\sqrt{\eta}H^{(1)}_{\beta+\frac{1}{2}}(k\eta) + C_2\sqrt{\eta}H^{(2)}_{\beta+\frac{1}{2}}(k\eta)$$

We choose following solution

$$\begin{cases} f_{\vec{k}} = \frac{\sqrt{\pi}}{2}\sqrt{\eta}H^{(1)}_{\beta+\frac{1}{2}}(k\eta) \\ f^*_{\vec{k}} = \frac{\sqrt{\pi}}{2}\sqrt{\eta}H^{(2)}_{\beta+\frac{1}{2}}(k\eta)\end{cases} \quad \text{it satisfies} \quad \begin{cases}\mathcal{W}_f = \dot{f}^*f - f^*\dot{f} = i \\ f \xrightarrow[\eta\to-\infty]{} \frac{e^{-ik\eta}}{\sqrt{2k}}\end{cases}$$

Vacuum solution:

$$\Psi_{\vec{k},0}(v_k) = \frac{1}{(2\pi)^{1/4}}\frac{1}{\sqrt{f^*}}e^{\frac{i}{2}\frac{\dot{f}^*}{f^*}v_k^2}$$

Knowing the classical solution, we automatically obtain vacuum solution $\Psi_{\vec{k},0}(v_k) = \frac{1}{(2\pi)^{1/4}}\frac{1}{\sqrt{f^*}}e^{\frac{i}{2}\frac{\dot{f}^*}{f^*}v_k^2}$.

Various classical solutions are parametrized by β, which is defined by the initial conditions. Lets recall that the wave function is a WKB wave-package consisting of a bundle of classical trajectories, i.e.

$$\begin{aligned}\Psi|_{S_0=const} &= \mathcal{A}(\beta)\prod_{\vec{k}}\Psi_{S,\vec{k}}(v_{S,\vec{k}}|\beta)\Psi_{T,\vec{k}}(v_{T,\vec{k}}|\beta) = \\ &= \mathcal{A}(\beta)\prod_{\vec{k}}\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{f^*}e^{\frac{i}{2}\frac{\dot{f}^*}{f^*}\left(v^2_{S,k}+\left(v^{(+)}_{T,k}\right)^2+\left(v^{(\times)}_{T,k}\right)^2\right)}\end{aligned}$$

Corresponding density matrix:

$$\rho(\beta,\tilde{\beta},\{v_{s,\vec{k}},v_{T,\vec{k}}\},\{\tilde{v}_{s,\vec{k}},\tilde{v}_{T,\vec{k}}\}) = \prod_{\vec{k},\vec{p}}\mathcal{A}(\beta)\mathcal{A}^*(\tilde{\beta})\psi_{\vec{k}}(\beta,v_{\vec{k}})\psi^*_{\vec{p}}(\tilde{\beta},v_{\vec{p}}) =$$
$$= \mathcal{A}(\beta)\mathcal{A}^*(\tilde{\beta})\prod_{\vec{k},\vec{p}}\frac{1}{2\pi}\frac{e^{\frac{i}{2}\frac{\dot{f}^*(\beta)}{f^*(\beta)}\left(v^2_{S,k}+\left(v^{(+)}_{T,k}\right)^2+\left(v^{(\times)}_{T,k}\right)^2\right)-\frac{i}{2}\frac{\dot{f}(\tilde{\beta})}{f(\tilde{\beta})}\left(v^2_{S,p}+\left(v^{(+)}_{T,p}\right)^2+\left(v^{(\times)}_{T,p}\right)^2\right)}}{f^*(\beta)f(\tilde{\beta})}$$

We now take into account the fact that conformal time is also a comoving horizon that defines a cause-related region. If the wavelength of the perturbation is more than the size of the horizon, then we cannot observe it in any way. Therefore, we are going to assume that these modes are a kind of environment for all the others, and we introduce a reduced density matrix, which allows us to observe the decoherence process depending on different β and $\tilde{\beta}$.

The reduced density matrix for short wavelength modes

$$\begin{aligned}\rho^{red} &= A(\beta)A^*(\tilde{\beta})\prod_{k<\frac{1}{|\eta|}}\left(\frac{i}{\dot{f}^*(\beta)f(\tilde{\beta}) - \dot{f}(\tilde{\beta})f^*(\beta)}\right)^{\frac{3}{2}} \times \\ &\times \prod_{k>\frac{1}{|\eta|}}\frac{1}{2\pi}\frac{1}{f(\tilde{\beta})f^*(\beta)}\cdot e^{-\frac{i}{2}\left(\frac{\dot{f}}{f}-\frac{\dot{f}^*}{f^*}\right)\left(v^2_{s,\vec{k}}+\left(v^{(+)}_{T,\vec{k}}\right)^2+\left(v^{(\times)}_{T,\vec{k}}\right)^2\right)}\end{aligned}$$

Until now, we have been considering the discrete spectrum of the wave vectors $\vec{k}$, but since the volume of space is actually infinite, then we can consider the spectrum as a continuous one. To do this, we rewrite the first product in the following way:

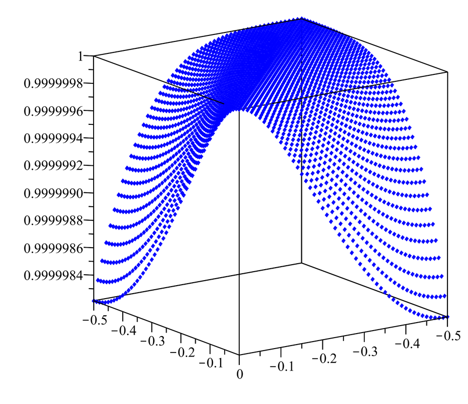
$$\prod_{k<\frac{1}{|\eta|}}\left(\frac{i}{\dot{f}^*(\beta)f(\tilde{\beta}) - \dot{f}(\tilde{\beta})f^*(\beta)}\right)^{\frac{3}{2}} = e^{\sum\limits_{k<\frac{1}{|\eta|}}\frac{3}{2}\ln\left(\frac{i}{\dot{f}^*(\beta)f(\tilde{\beta})-\dot{f}(\tilde{\beta})f^*(\beta)}\right)} =$$
$$= e^{\sum\limits_{k<\frac{1}{|\eta|}}-\frac{3}{2}\ln\left((-i)\dot{f}^*(\beta)f(\tilde{\beta})-\dot{f}(\tilde{\beta})f^*(\beta)\right)}$$

The sum in the exponent can be replaced back by the integral

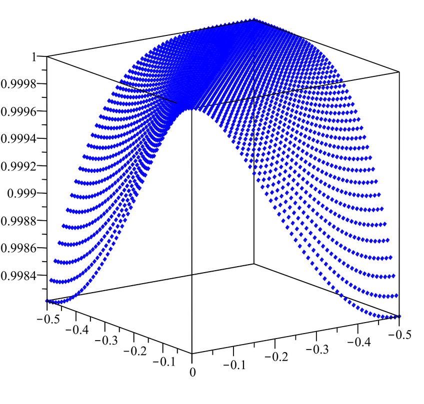
$$\sum_{k<\frac{1}{|\eta|}}\{\ldots\} \to \mathscr{L}^3\int_{\frac{1}{\eta}}^{0}d^3k\{\ldots\}$$

Here we have taken into account the fact that the conformal time takes only negative values.

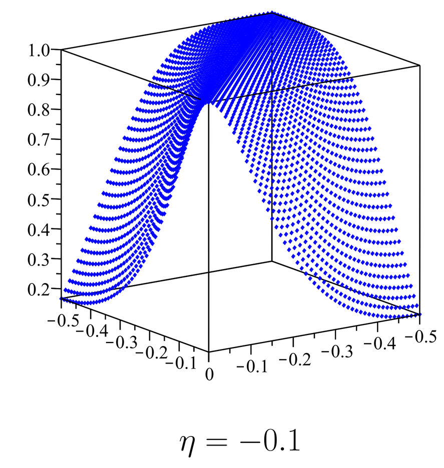
Due to the complexity of the integrand, the integration was performed numerically. As a result, we obtained the following graphs of the short-wave part of the density matrix for different values η.



$\eta = -10$



$\eta = -1$



$\eta = -0.1$

The graphs clearly show that the diagonal is normalized to 1; as the difference between $\beta - \tilde{\beta}$ increases, the off-diagonal terms decrease. With a decrease in η modulo, that is, when the system tends to $+\infty$ in cosmological time, the diagonal elements increasingly dominates, so the system behaves in an increasingly classical way, and the interference is suppressed.

4. Conclusion

- We considered the evolution of the universe, which was originally in a vacuum state.
- Over time, an increasing number of perturbation modes of the metric and inflaton field goes beyond the cosmological horizon and becomes unobservable. We showed that during this process, the density matrix becomes mixed for short wavelength modes, in the limit diagonal in the β, the variable characterizing the rate of inflation.
- This suggests that, from the point of view of the local observer, the coherent wave package in late times corresponds to the classical probability distribution of background metrics.

References

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