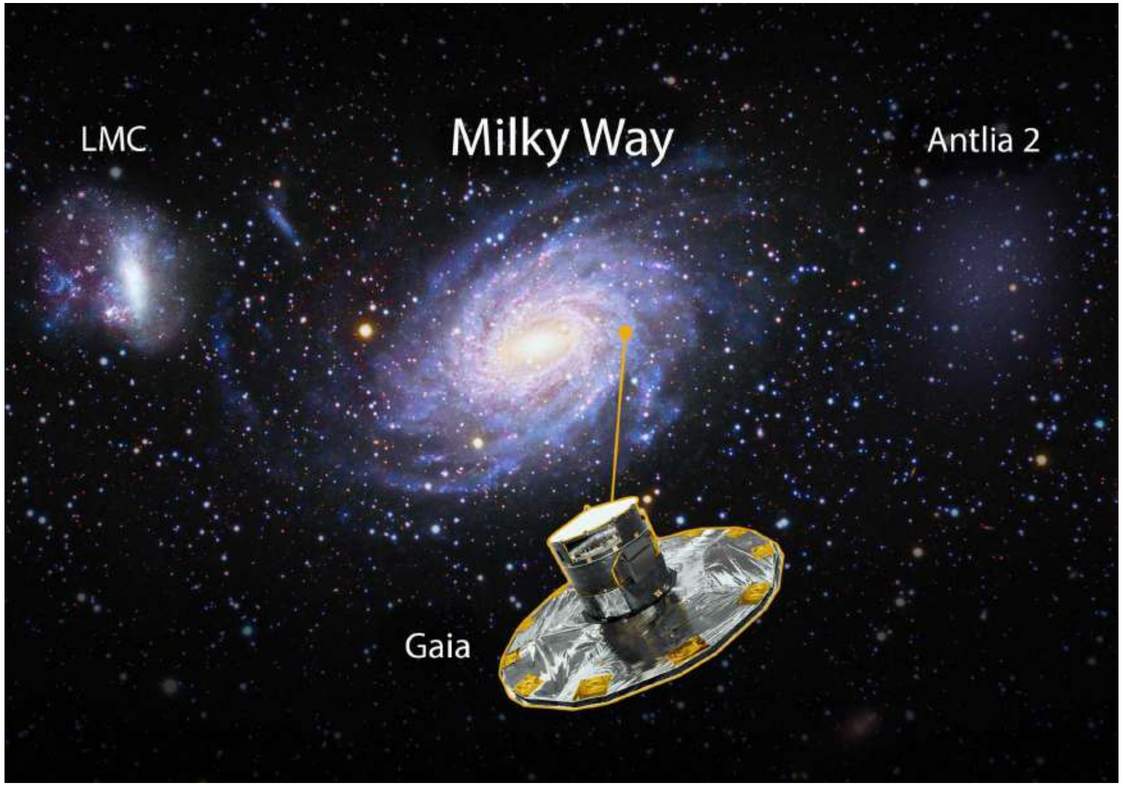
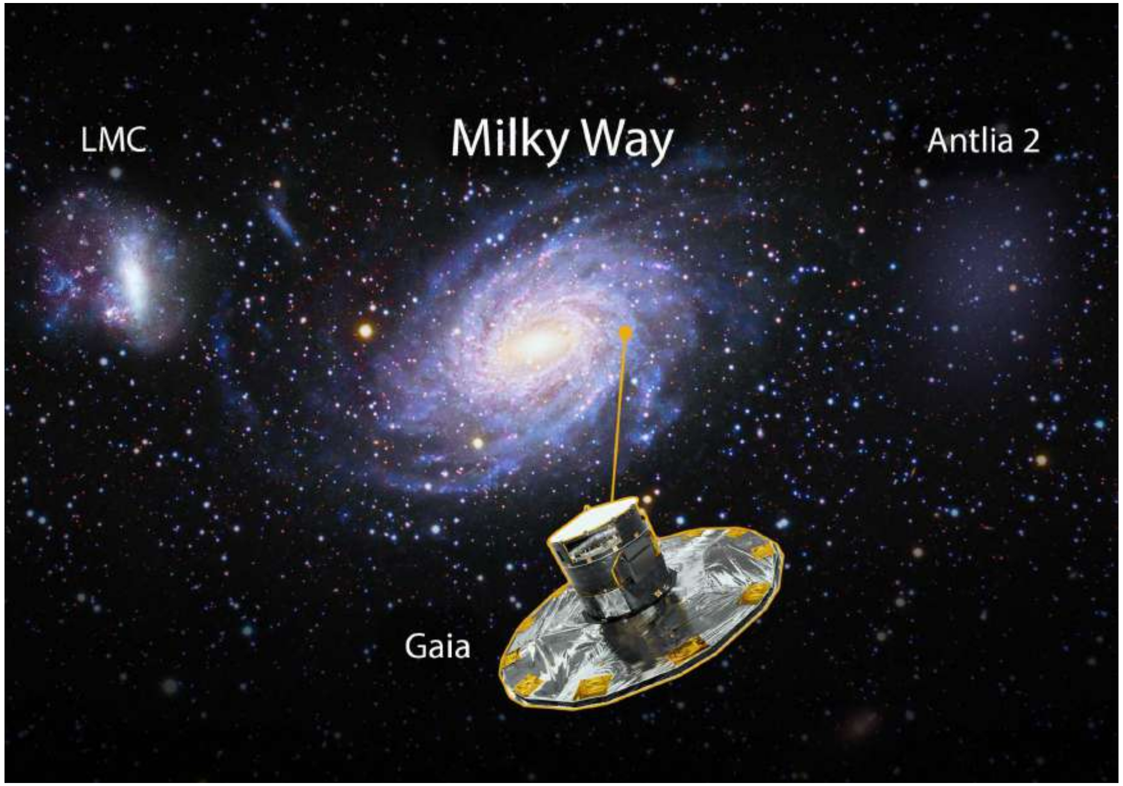
Tasks for seminar

1. Estimate the arrival time delay of mass eigenstates of an electron neutrino with energy 15 MeV born in SN 1987A, assuming $m_1 = 0$, $m_2 = 8.6$, $m_3 = 50$ meV.



Inputs: Distance from LMC is about 50 kps (experimental range is 40–55 kps), 1 ps $\approx 30.9 \times 10^{12}$ km.

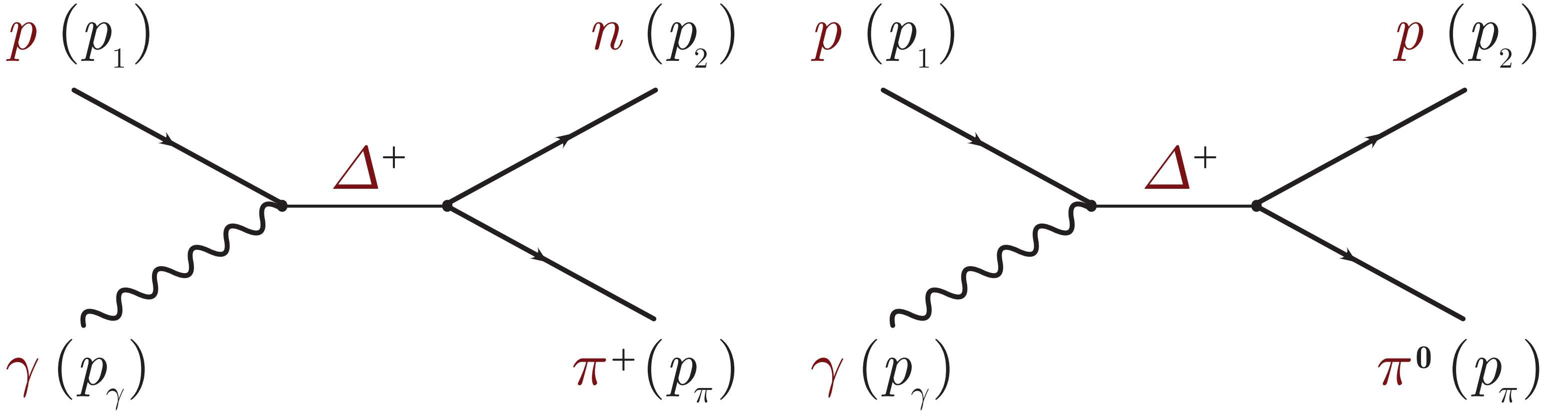
1. Estimate the arrival time delay of mass eigenstates of an electron neutrino with energy 15 MeV born in SN 1987A, assuming $m_1 = 0$, $m_2 = 8.6$, $m_3 = 50$ meV.



Inputs: Distance from LMC is about 50 kps (experimental range is 40–55 kps), 1 ps $\approx 30.9 \times 10^{12}$ km.

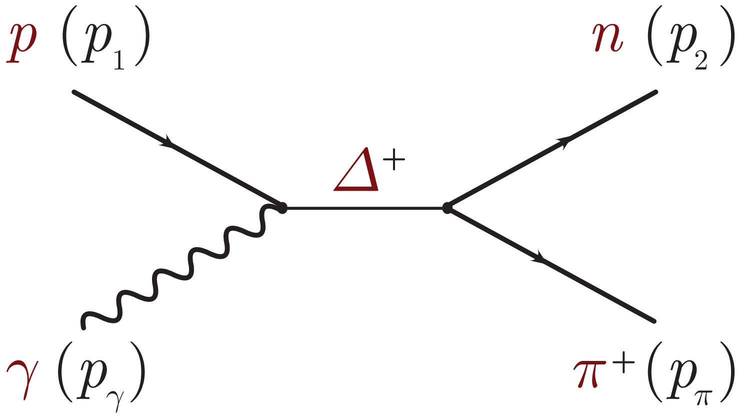
Solution: $\delta t_{1i} \approx \dfrac{m_i^2}{2E_\nu^2}\dfrac{L_{\text{LMC}}}{c} \implies \delta t_{12} \approx 8.5 \times 10^{-7}$ s, $\delta t_{13} \approx 2.9 \times 10^{-5}$ s.

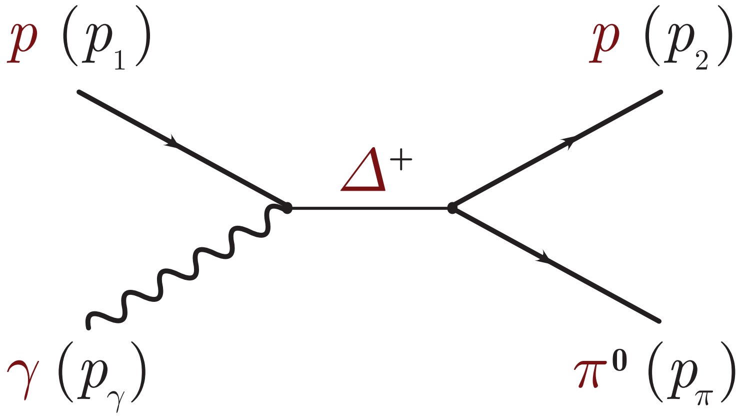
2. Estimate the pion production threshold in a collision of a CR proton with a CMB photon.



Inputs: $\langle E_\gamma \rangle_{\text{CMB}} \equiv \langle h\nu_\gamma \rangle = k_B T_{\text{CMB}} \simeq 2.349 \times 10^{-4}$ eV, $m_{\pi^+} \simeq 139.56995$ MeV, $m_{\pi^0} \simeq 134.97660$ MeV, $m_p \simeq 938.27231$ MeV, $m_n \simeq 939.56536$ MeV.

2. Estimate the pion production threshold in a collision of a CR proton with a CMB photon.





Inputs: $\langle E_\gamma \rangle_{\text{CMB}} \equiv \langle h\nu_\gamma \rangle = k_B T_{\text{CMB}} \simeq 2.349 \times 10^{-4}$ eV, $m_{\pi^+} \simeq 139.56995$ MeV, $m_{\pi^0} \simeq 134.97660$ MeV, $m_p \simeq 938.27231$ MeV, $m_n \simeq 939.56536$ MeV.

Solution: $s = (p_1 + p_\gamma)^2 = m_p^2 + 2E_\gamma (E_1 - P_1 \cos\theta)$, where $E_1^2 = P_1^2 + m_p^2$, $P_1 = |\mathbf{p}_1|$. On the other hand, $s = (p_2 + p_\pi)^2 = (p_2^* + p_\pi^*)^2$, where $*$ marks the center of mass frame of the final state $(\mathbf{p}_1^* + \mathbf{p}_\pi^* = 0)$ $\Longrightarrow$ $s = (E_2^* + E_\pi^*)^2 \geq (m_N + m_\pi)^2$ $\Longrightarrow$ $2E_\gamma (E_1 - P_1 \cos\theta) \geq (m_N + m_\pi)^2 - m_p^2$. Clearly $P_1^2 \gg m_p^2$ $\Longrightarrow$

$$E_{\text{th}} = E_1\big|_{\theta=\pi} \simeq \frac{(m_N + m_\pi)^2 - m_p^2}{4E_\gamma} \simeq \begin{cases} 3.0 \times 10^{20} \dfrac{\langle h\nu_\gamma \rangle}{E_\gamma} \text{ eV for } p\gamma \to n\pi^+, \\ 2.9 \times 10^{20} \dfrac{\langle h\nu_\gamma \rangle}{E_\gamma} \text{ eV for } p\gamma \to p\pi^0. \end{cases}$$

3. Estimate the maximum energy of the neutrino from a GZK pion.

Inputs: $m_\pi \simeq 139.569950$ MeV, $m_\mu \simeq 105.658387$ MeV, $m_e \simeq 0.51099907$ MeV.

3. Estimate the maximum energy of the neutrino from a GZK pion.

Inputs: $m_\pi \simeq 139.569950$ MeV, $m_\mu \simeq 105.658387$ MeV, $m_e \simeq 0.51099907$ MeV.

Solution: $E_\nu^* = \dfrac{m_\pi^2 - m_\ell^2}{2m_\pi} \Longrightarrow E_\nu = \mathit{\Gamma}\left(E_\nu^* - \mathbf{v}\mathbf{p}_\nu^*\right) \Longrightarrow E_\nu^{\max} \approx \left(1 - \dfrac{m_\ell^2}{m_\pi^2}\right) E_\pi$
$\Longrightarrow E_\nu^{\max} \approx 0.42691 E_\pi$ for ν_μ and $0.999987 E_\pi$ for ν_e.

4. Prove that any nonsingular matrix $\mathbf{M}$ can be diagonalized by a bi-unitary transformation

$$\mathbf{M} = \widetilde{\mathbf{V}}\mathbf{m}\mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k\delta_{kl}|| = \mathsf{diag}\,(m_1, m_2, \ldots, m_N)\,, \quad m_k > 0, \quad \mathbf{V}\mathbf{V}^\dagger = \widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger = \mathbf{1}.$$

Comment: Recall that this theorem plays an important role in the theory of the Dirac neutrino.

4. Prove that any nonsingular matrix $\mathbf{M}$ can be diagonalized by a bi-unitary transformation

$$\mathbf{M} = \widetilde{\mathbf{V}}\mathbf{m}\mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k\delta_{kl}|| = \text{diag}\left(m_1, m_2, \ldots, m_N\right), \quad m_k > 0, \quad \mathbf{V}\mathbf{V}^\dagger = \widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger = \mathbf{1}.$$

Comment: Recall that this theorem plays an important role in the theory of the Dirac neutrino.

Proof: Matrix $\mathbf{M}\mathbf{M}^\dagger$ is Hermitian, $\left(\mathbf{M}\mathbf{M}^\dagger\right)^\dagger = \mathbf{M}\mathbf{M}^\dagger$, $\Longrightarrow$ there exist a unitary matrix $\widetilde{\mathbf{V}}$ such that

$$\widetilde{\mathbf{V}}^\dagger\left(\mathbf{M}\mathbf{M}^\dagger\right)\widetilde{\mathbf{V}} = \mathbf{m}^2 = \text{diag}\left(m_1^2, m_2^2, \ldots, m_N^2\right),$$

where $m_i^2 > 0$ for any i. Indeed, $\mathbf{M}^\dagger\widetilde{\mathbf{V}} = \left(\widetilde{\mathbf{V}}^\dagger\mathbf{M}\right)^\dagger$ and thus

$$m_i^2 = \sum_j \left(\widetilde{\mathbf{V}}^\dagger\mathbf{M}\right)_{ij}\left(\widetilde{\mathbf{V}}^\dagger\mathbf{M}\right)_{ij}^* = \sum_j \left|\left(\widetilde{\mathbf{V}}^\dagger\mathbf{M}\right)_{ij}\right|^2 \geq 0;$$

the equality is however excluded since $\mathbf{m}^2$ is nonsingular. Let's now define the matrix $\mathbf{V} = \mathbf{M}^\dagger\widetilde{\mathbf{V}}\mathbf{m}^{-1}$. We have:

$$\mathbf{V}^\dagger = \mathbf{m}^{-1}\widetilde{\mathbf{V}}^\dagger\mathbf{M} \Longrightarrow \mathbf{V}^\dagger\mathbf{V} = \mathbf{m}^{-1}\widetilde{\mathbf{V}}^\dagger\mathbf{M}\mathbf{M}^\dagger\widetilde{\mathbf{V}}\mathbf{m}^{-1} = \mathbf{1},$$

that is the matrix $\mathbf{V}$ is unitary and $\widetilde{\mathbf{V}}^\dagger\mathbf{M}\mathbf{V} = \mathbf{m}$.

5. Find the masses of physical neutrinos for the Lagrangian with a mass matrix

$$\mathbf{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}, \quad (m_{L,R,D} > 0).$$

Comment: Recall that this trivial example is the basis for the see-saw mechanism.

5. Find the masses of physical neutrinos for the Lagrangian with a mass matrix

$$\mathbf{M} = \begin{pmatrix} M_L & M_1 \\ M_2 & M_R \end{pmatrix} \quad (M_{L,R,1,2} \geq 0).$$

Comment: Recall that this trivial example is the basis for the see-saw mechanism.

Solution: Eigenvalues $m_{1,2}$ of the matrix $\mathbf{M}$ satisfy the equation $\det(\lambda - \mathbf{M}) = 0$. Therefore $\lambda^2 - (M_L + M_R)\,\lambda + M_L M_R - M_1 M_2 = 0$. The solution is

$$\lambda_\pm = \frac{1}{2}\left[M_L + M_R \pm \sqrt{(M_L - M_R)^2 + 4M_1 M_2}\right].$$

Note: λ_- can be negative if $M_1 M_2 > M_L M_R$. Since, however, the sign of the eigenfields can always be redefined, the physical masses are $m_1 = \lambda_+$ and $m_2 = |\lambda_-|$.

Let's now try to diagonalize $\mathbf{M}$ by a unitary transformation

$$\mathbf{V}^\dagger \mathbf{M} \mathbf{V} = \text{diag}(\lambda_-, \lambda_+) \equiv \mathbf{m}. \tag{1}$$

Since the $\mathbf{M}$ is positive definite, $\mathbf{V}^\dagger = \mathbf{V}^\mathsf{T} \Longrightarrow \mathbf{V}$ is just a rotation matrix,

$$\mathbf{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Longrightarrow \mathbf{V}^\mathsf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{V}\mathbf{V}^\mathsf{T} = \mathbf{1}.$$

From Eq. (1) we have

$$\mathbf{VmV}^\dagger = \mathbf{M} \implies \begin{pmatrix} \cos^2\theta\,\lambda_- + \sin^2\theta\,\lambda_+ & \sin\theta\cos\theta\,(\lambda_+ - \lambda_-) \\ \sin\theta\cos\theta\,(\lambda_- + \lambda_-) & \cos^2\theta\,\lambda_+ + \sin^2\theta\,\lambda_- \end{pmatrix} = \begin{pmatrix} M_L & M_1 \\ M_2 & M_R \end{pmatrix}.$$

Oh, the horror! We got $\sin\theta\cos\theta\,(\lambda_+ - \lambda_-) = M_1$ and $\sin\theta\cos\theta\,(\lambda_- + \lambda_-) = M_2$. What does that mean?! Nothing unexpected. The Majorana mass matrix should be symmetric, otherwise the unitary transformation we need does not exist. So further we put $M_1 = M_2 = M_D$. The order of the eigenvalues in Eq. (1) provides $\theta > 0$. We have

$$\left(\cos^2\theta - \sin^2\theta\right)(\lambda_+ - \lambda_-) = M_R - M_L \quad \text{and} \quad \sin\theta\cos\theta\,(\lambda_+ - \lambda_-) = M_D.$$

Given that $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$ we obtain

$$\tan 2\theta = \frac{2M_D}{M_R - M_L} \iff \theta = \frac{1}{2}\arctan\left(\frac{2M_D}{M_R - M_L}\right).$$

Let's now consider the most interesting special case $M_R \equiv M \gg M_D \equiv m$ and $M_L = 0$. Then

$$\lambda_+ \simeq M, \quad \lambda_- \simeq -\frac{m^2}{M} \simeq -\theta m, \quad \text{and} \quad \theta \simeq \frac{m}{M}.$$

This is the see-saw case: $m_1 = \lambda_+$ is a large (GUT?) mass and $m_2 = -\lambda_-$ is a small mass.