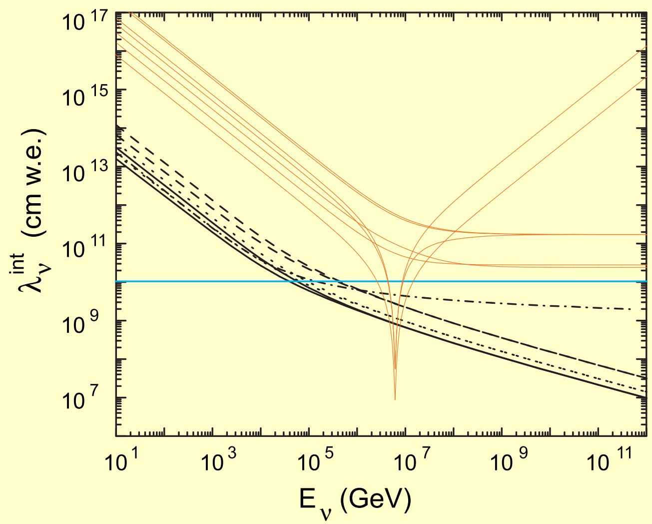
Neutrino oscillations in matter



6 Neutrino refraction.

It has been noted by Wolfenstein[a] that neutrino oscillations in a medium are affected by interactions even if the thickness of the medium is negligible in comparison with the neutrino mean free path.

Let us forget for the moment about the inelastic collisions and consider the simplest case of a ultrarelativistic neutrino which moves in an external (effective) potential W formed by the matter background. If the neutrino momentum in vacuum was $\mathbf{p}$ then its energy was $\simeq p = |\mathbf{p}|$. When the neutrino enters into the medium, its energy becomes $E = p + W$. Let us now introduce the index of refraction $n = p/E$ which is a positive value in the absence of inelastic collisions. Therefore

$$W = (1 - n)E \simeq (1 - n)p. \tag{18}$$

In the last step, we took into account that neutrino interaction with matter is very weak, $|W| \ll E$, and thus $E \simeq p$ is a good approximation.

The natural generalization of Eq. (13) for the time evolution of neutrino flavor states in matter then follows from this simple consideration and the quantum-mechanical correspondence principle.

[a] L. Wolfenstein, Phys. Rev. D **17** (1978) 2369.

This is the famous Wolfenstein equation:

$$i\frac{d}{dt}|\boldsymbol{\nu}(t)\rangle_f = \left[\mathbf{V}\mathbf{H}_0\mathbf{V}^\dagger + \mathbf{W}(t)\right]|\boldsymbol{\nu}(t)\rangle_f, \tag{19}$$

where

$$\mathbf{W}(t) = \text{diag}\left(1 - n_{\nu_e}, 1 - n_{\nu_\mu}, 1 - n_{\nu_\tau}, \ldots\right) p \tag{20}$$

is the interaction Hamiltonian.

It will be useful for the following to introduce the *time-evolution operator* for the flavor states defined by

$$|\boldsymbol{\nu}(t)\rangle_f = \mathbf{S}(t)|\boldsymbol{\nu}(0)\rangle_f.$$

Taking into account that $|\boldsymbol{\nu}(t)\rangle_f$ must satisfy Eq. (19) for any initial condition $|\boldsymbol{\nu}(t=0)\rangle_f = |\boldsymbol{\nu}(0)\rangle_f$, the Wolfenstein equation can be immediately rewritten in terms of the evolution operator:

$$i\dot{\mathbf{S}}(t) = \left[\mathbf{V}\mathbf{H}_0\mathbf{V}^\dagger + \mathbf{W}(t)\right]\mathbf{S}(t), \quad \mathbf{S}(0) = \mathbf{1}. \tag{21}$$

This equation (or its equivalent (19)) cannot be solved analytically in the general case of a medium with a varying (along the neutrino pass) density. But for a medium with a slowly (adiabatically) varying density distribution the approximate solution can be obtained by a diagonalization of the effective Hamiltonian. Below we will consider this method for a rather general 2-flavor case but now let us illustrate (without derivation) the simplest situation with a matter of constant density.

6.1 Matter of constant density.

In the 2-flavor case, the transition probability is given by the formula very similar to that for vacuum:

$$P_{\alpha\alpha'}(L) = \frac{1}{2}\sin^2 2\theta_{\mathsf{m}}\left[1-\cos\left(\frac{2\pi L}{L_{\mathsf{m}}}\right)\right],$$

$$L_{\mathsf{m}} = L_{\mathsf{v}}\left[1-2\kappa\left(L_{\mathsf{v}}/L_0\right)\cos 2\theta+\left(L_{\mathsf{v}}/L_0\right)^2\right]^{-1/2}.$$

The L_{m} is called the oscillation length in matter and is defined through the following quantities:

$$L_{\mathsf{v}} \equiv L_{23} = \frac{4\pi E}{\Delta m^2}, \quad L_0 = \frac{\sqrt{2}\pi A}{G_F N_A Z \rho} \approx 2R_{\oplus}\left(\frac{A}{2Z}\right)\left(\frac{2.5\ \mathsf{g/cm}^3}{\rho}\right),$$

$$\kappa = \mathsf{sign}\left(m_3^2-m_2^2\right), \quad \Delta m^2 = \left|m_3^2-m_2^2\right|.$$

The parameter θ_{m} is called the mixing angle in matter and is given by

$$\sin 2\theta_{\mathsf{m}} = \sin 2\theta\left(\frac{L_{\mathsf{m}}}{L_{\mathsf{v}}}\right),$$

$$\cos 2\theta_{\mathsf{m}} = \left(\cos 2\theta - \kappa\frac{L_{\mathsf{v}}}{L_0}\right)\left(\frac{L_{\mathsf{m}}}{L_{\mathsf{v}}}\right).$$

The solution for antineutrinos is the same but with the replacement

$$\kappa \longmapsto -\kappa.$$

The closeness of the value of L_0 to the Earth's diameter is even more surprising than that for L_{v}. The matter effects are therefore important for atmospheric neutrinos.

7 Propagation of high-energy mixed neutrinos through matter.

"The matter doesn't matter"

Lincoln Wolfenstein, lecture given at 28th
SLAC Summer Institute on Particle Physics
"Neutrinos from the Lab, the Sun, and the
Cosmos", Stanford, CA, Aug. 14-25, 2000.

When neutrinos propagate through vacuum there is a phase change $\exp\left(-im_i^2 t/2p_\nu\right)$. For two mixed flavors there is a resulting oscillation with length

$$L_{\text{vac}} = \frac{4\pi E_\nu}{\Delta m^2} \approx D_\oplus \left(\frac{E_\nu}{10\ \text{GeV}}\right)\left(\frac{0.002\ \text{eV}^2}{\Delta m^2}\right).$$

In matter there is an additional phase change due to refraction associated with forward scattering $\exp\left[ip_\nu(\text{Re}\, n - 1)t\right]$.
The characteristic length (for a normal medium) is

$$L_{\text{ref}} = \frac{\sqrt{2}A}{G_F N_A Z\rho} \approx D_\oplus \left(\frac{A}{2Z}\right)\left(\frac{2.5\ \text{g/cm}^2}{\rho}\right).$$

It is generally believed that the imaginary part of the index of refraction n which describes the neutrino absorption due to inelastic interactions *does not affect the oscillation probabilities* or at the least inelastic interactions can be someway *decoupled* from oscillations.

The conventional arguments are

- $\mathrm{Re}\, n - 1 \propto G_F$ while $\mathrm{Im}\, n \propto G_F^2$;
- Only Δn may affect the oscillations and $\Delta \mathrm{Im}\, n$ is all the more negligible.

It will be shown that these arguments do not work for sufficiently high neutrino energies and/or for thick media $\Longrightarrow$ in general absorption cannot be decoupled from refraction and mixing.[a] By using another cant phrase of Wolfenstein, one can say that

"In some circumstances the matter could matter."

7.1 Generalized MSW equation.

Let

$f_{\nu_\alpha A}(0)$ be the amplitude for the ν_α zero-angle scattering from particle A of the matter background ($A = e, p, n, \ldots$),

$\rho(t)$ be the matter density (in g/cm^3),

$Y_A(t)$ be the number of particles A per amu in the point t of the medium, and

$N_0 = 6.02214199 \times 10^{23}$ cm^{-3} be the reference particle number density (numerically equal to Avogadro's number).

Then the index of refraction of ν_α for small $|n-1|$ (for normal media $|n-1| \lll 1$) is given by

$$n_\alpha(t) = 1 + \frac{2\pi N_0 \rho(t)}{p_\nu^2} \sum_A Y_A(t) f_{\nu_\alpha A}(0),$$

where p_ν is the neutrino momentum.

[a] $p_\nu \mathrm{Im}\, n \propto \sigma^{\text{tot}}(p_\nu)$ grows fast with energy while $p_\nu\, (\mathrm{Re}\, n - 1)$ is a constant or decreasing function of E_ν.

Since the amplitude $f_{\nu_\alpha A}(0)$ is in general a complex number, the index of refraction is also complex. Its real part is responsible for neutrino refraction while the imaginary part – for absorption. From the optical theorem of quantum mechanics we have

$$\mathsf{Im}\left[f_{\nu_\alpha A}(0)\right] = \frac{p_\nu}{4\pi}\sigma^{\mathsf{tot}}_{\nu_\alpha A}\left(p_\nu\right).$$

This implies that

$$p_\nu \mathsf{Im}\left[n_\alpha(t)\right] = \frac{1}{2}N_0\rho(t)\sum_A Y_A(t)\sigma^{\mathsf{tot}}_{\nu_\alpha A}\left(p_\nu\right) = \frac{1}{2\Lambda_\alpha\left(p_\nu, t\right)},$$

where

$$\Lambda_\alpha\left(p_\nu, t\right) = \frac{1}{\Sigma^{\mathsf{tot}}_\alpha\left(p_\nu, t\right)} = \frac{\lambda^{\mathsf{tot}}_a\left(p_\nu, t\right)}{\rho(t)}.$$

is the mean free path [in cm] of ν_α in the point t of the medium. Since the neutrino momentum, p_ν, is an extrinsic variable in Eq. (22), we will sometimes omit this argument to simplify formulas.

The generalized MSW equation for the time-evolution operator

$$\mathbf{S}(t) = \begin{pmatrix} S_{\alpha\alpha}(t) & S_{\alpha\beta}(t) \\ S_{\beta\alpha}(t) & S_{\beta\beta}(t) \end{pmatrix}$$

of two mixed stable neutrino flavors ν_α and ν_β propagating through an absorbing medium can be written as

$$\boxed{i\frac{d}{dt}\mathbf{S}(t) = \left[\mathbf{V}\mathbf{H}_0\mathbf{V}^T + \mathbf{W}(t)\right]\mathbf{S}(t), \quad \left(\mathbf{S}(0) = \mathbf{1}\right).} \tag{22}$$

Here

$$\mathbf{V}= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ is the vacuum mixing matrix ($0 \le \theta \le \pi/2$),

$$\mathbf{H}_0= \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$ is the vacuum Hamiltonian for ν mass eigenstates,

$$E_i= \sqrt{p_\nu^2 + m_i^2} \simeq p_\nu + m_i^2/2p_\nu$$ is the energy of the ν_i eigenstate,

$$\mathbf{W}(t)= -p_\nu \begin{pmatrix} n_\alpha(t) - 1 & 0 \\ 0 & n_\beta(t) - 1 \end{pmatrix}$$ is the interaction Hamiltonian.

7.2 Master equation.

It is useful to transform MSW equation into the one with a traceless Hamiltonian. For this purpose we define the matrix

$$\widetilde{\mathbf{S}}(t) = \exp\left\{\frac{i}{2}\int_0^t \mathrm{Tr}\left[\mathbf{H}_0 + \mathbf{W}(t')\right] dt'\right\} \mathbf{S}(t).$$

The master equation (ME) for this matrix then is

$$\boxed{i\frac{d}{dt}\widetilde{\mathbf{S}}(t) = \mathbf{H}(t)\widetilde{\mathbf{S}}(t), \quad \widetilde{\mathbf{S}}(0) = \mathbf{1}.} \tag{23}$$

The effective Hamiltonian is defined by

$$\mathbf{H}(t) = \begin{pmatrix} q(t) - \Delta_c & \Delta_s \\ \Delta_s & -q(t) + \Delta_c \end{pmatrix},$$

$$\Delta_c = \Delta \cos 2\theta, \quad \Delta_s = \Delta \sin 2\theta, \quad \Delta = \frac{m_2^2 - m_1^2}{4p_\nu},$$

$$q(t) = q_R(t) + iq_I(t) = \frac{1}{2} p_\nu \left[n_\beta(t) - n_\alpha(t) \right].$$

The Hamiltonian for antineutrinos is of the same form as $\mathbf{H}(t)$ but

$$\mathrm{Re}\left[f_{\overline{\nu}_\alpha A}(0)\right] = -\mathrm{Re}\left[f_{\nu_\alpha A}(0)\right] \quad \text{and} \quad \mathrm{Im}\left[f_{\overline{\nu}_\alpha A}(0)\right] \neq \mathrm{Im}\left[f_{\nu_\alpha A}(0)\right].$$

The neutrino oscillation probabilities are

$$P\left[\nu_\alpha(0) \to \nu_{\alpha'}(t)\right] \equiv P_{\alpha\alpha'}(t) = \left|S_{\alpha'\alpha}(t)\right|^2 = A(t) \left|\widetilde{S}_{\alpha'\alpha}(t)\right|^2, \tag{24}$$

where

$$A(t) = \exp\left[-\int_0^t \frac{dt'}{\Lambda(t')} \right], \quad \frac{1}{\Lambda(t)} = \frac{1}{2} \left[\frac{1}{\Lambda_\alpha(t)} + \frac{1}{\Lambda_\beta(t)} \right].$$

Owing to the complex potential q, the Hamiltonian $\mathbf{H}(t)$ is non-Hermitian and the new evolution operator $\widetilde{\mathbf{S}}(t)$ is nonunitary. As a result, there are no conventional relations between $P_{\alpha\alpha'}(t)$.

Since

$$q_I(t) = \frac{1}{4}\left[\frac{1}{\Lambda_\beta(t)} - \frac{1}{\Lambda_\alpha(t)}\right],$$

the matrix $\mathbf{H}(t)$ becomes Hermitian when $\Lambda_\alpha = \Lambda_\beta$. If this is the case at any t, the ME reduces to the standard MSW equation and inelastic scattering results in the common exponential attenuation of the probabilities. From here, we shall consider the more general and more interesting case, when $\Lambda_\alpha \neq \Lambda_\beta$.

7.3 Examples.

$\boxed{\nu_\alpha - \nu_s}$

This is the extreme example. Since $\Lambda_s = \infty$, we have $\Lambda = 2\Lambda_\alpha$ and $q_I = -1/4\Lambda_\alpha$. So $q_I \neq 0$ at any energy. Even without solving the evolution equation, one can expect the penetrability of active neutrinos to be essentially modified in this case because, roughly speaking, they spend a certain part of life in the sterile state. In other words, sterile neutrinos "tow" their active companions through the medium as a tugboat. On the other hand, the active neutrinos "retard" the sterile ones, like a bulky barge retards its tugboat. As a result, the sterile neutrinos undergo some absorption.

$\boxed{\nu_{e,\mu} - \nu_\tau}$

Essentially at all energies, $\sigma^{\text{CC}}_{\nu_{e,\mu}N} > \sigma^{\text{CC}}_{\nu_\tau N}$. This is because of large value of the τ lepton mass, m_τ, which leads to several consequences:

1. high neutrino energy threshold for τ production;
2. sharp shrinkage of the phase spaces for CC $\nu_\tau N$ reactions;
3. kinematic correction factors ($\propto m_\tau^2$) to the nucleon structure functions (the corresponding structures are negligible for e production and small for μ production).

The neutral current contributions are canceled out from q_I. Thus, in the context of the master equation, ν_τ can be treated as (almost) sterile within the energy range for which $\sigma^{\text{CC}}_{\nu_{e,\mu}N} \gg \sigma^{\text{CC}}_{\nu_\tau N}$ (see Figures in pp. 109–110).

$\boxed{\overline{\nu}_e - \overline{\nu}_\alpha}$

A similar situation, while in quite a different and narrow energy range, holds in the case of mixing of $\overline{\nu}_e$ with some other flavor. This is a particular case for a normal C asymmetric medium, because of the W boson resonance formed in the neighborhood of $E_\nu^{\text{res}} = m_W^2/2m_e \approx 6.33$ PeV through the reactions

$$\overline{\nu}_e e^- \to W^- \to \text{hadrons} \quad \text{and} \quad \overline{\nu}_e e^- \to W^- \to \overline{\nu}_\ell \ell^- \quad (\ell = e, \mu, \tau).$$

Let's remind that $\sigma^{\text{tot}}_{\overline{\nu}_e e} \approx 250\ \sigma^{\text{tot}}_{\overline{\nu}_e N}$ just at the resonance peak.

7.4 Total cross sections.

According to Albright and Jarlskog[a]

$$\frac{d\sigma^{\mathsf{CC}}_{\nu,\overline{\nu}}}{dxdy} = \frac{G_F^2 m_N E_\nu}{\pi}\left(A_1F_1 + A_2F_2 \pm A_3F_3 + A_4F_4 + A_5F_5\right),$$

where $F_i = F_i(x,Q^2)$ are the nucleon structure functions and A_i are the kinematic factors $i = 1,\ldots,5$). These factors were calculated by many authors[b] and the most accurate formulas were given by Paschos and Yu:

$$A_1 = xy^2 + \frac{m_l^2 y}{2m_N E_\nu}, \quad A_2 = 1 - y - \frac{m_N}{2E_\nu}xy - \frac{m_l^2}{4E_\nu^2}, \quad A_3 = xy\left(1 - \frac{y}{2}\right) - \frac{m_l^2 y}{4m_N E_\nu},$$
$$A_4 = \frac{m_l^2}{2m_N E_\nu}\left(xy + \frac{m_l^2}{2m_N E_\nu}\right), \quad A_5 = -\frac{m_l^2}{2m_N E_\nu}.$$

The contributions proportional to m_ℓ^2 must vanish as $E_\nu \gg m_\ell$. However they remain surprisingly important even at very high energies.

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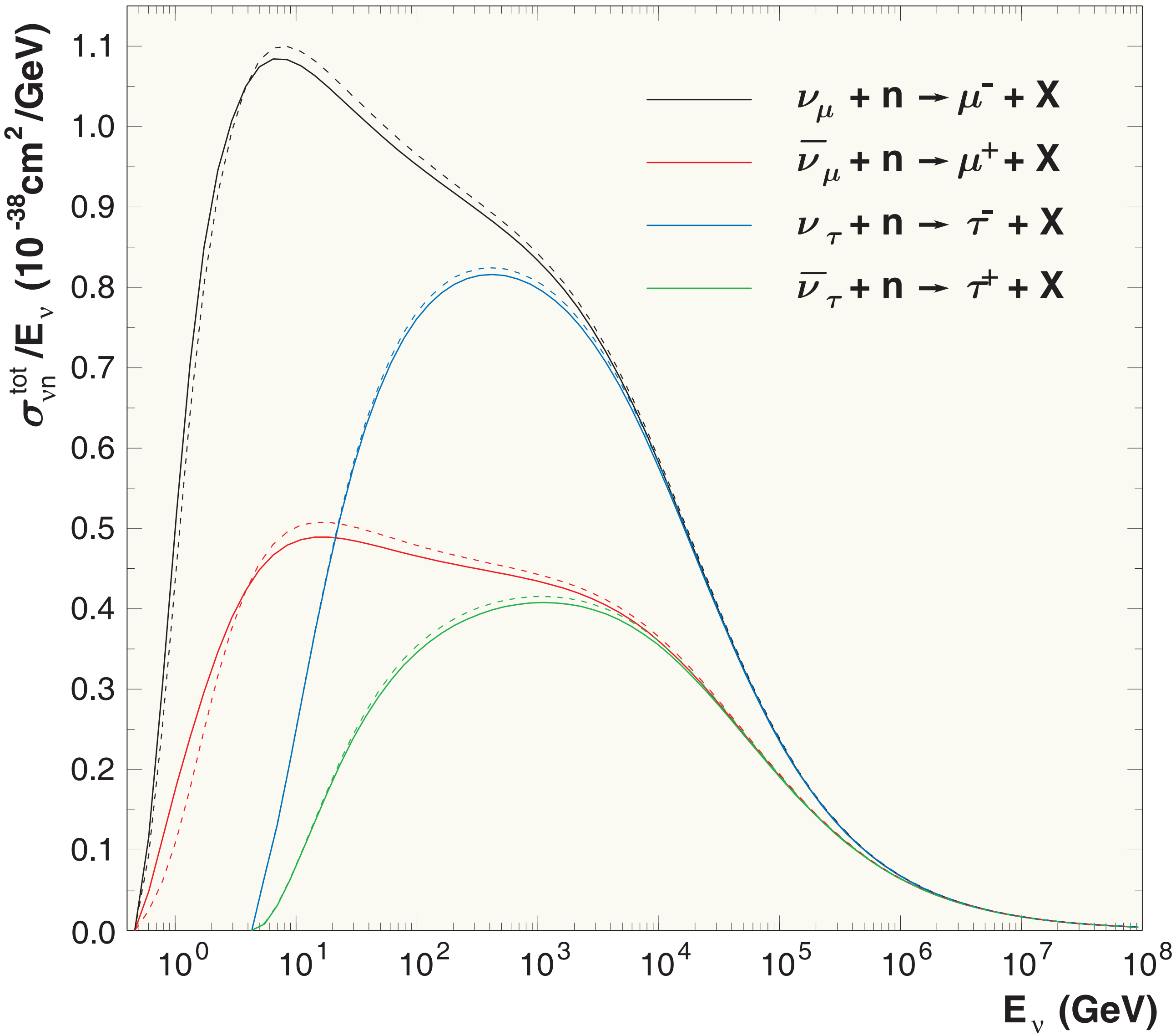
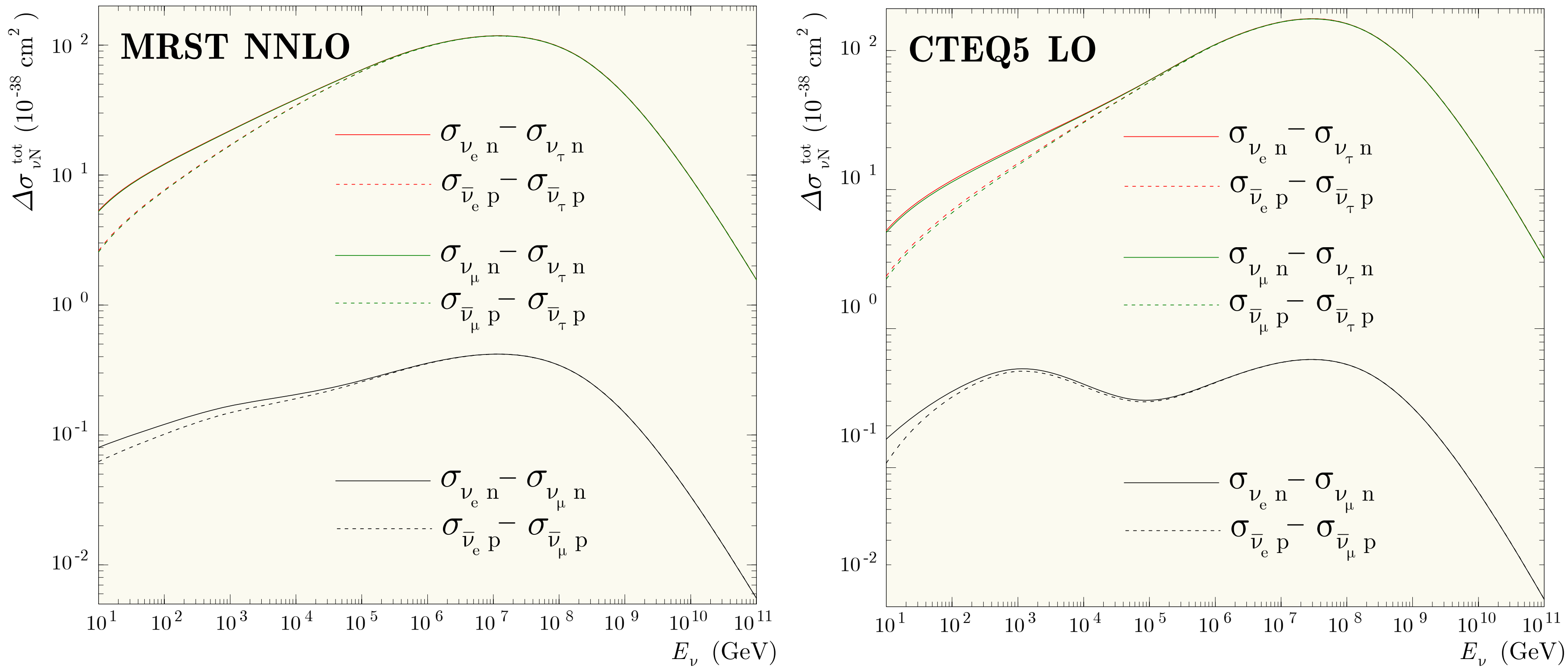
[a] C. H. Albright and C. Jarlskog, Nucl. Phys. B **84** (1975) 467–492; see also I. Ju, Phys. Rev. D **8** (1973) 3103–3109 and V. D. Barger *et al.*, Phys. Rev. D **16** (1977) 2141–2157.

[b] See previous footnote and also the more recent papers: S. Dutta, R. Gandhi, and B. Mukhopadhyaya, Eur. Phys. J. C **18** (2000) 405–416, hep-ph/9905475; N. I. Starkov, J. Phys. G **27** (2001) L81–L85; E. A. Paschos and J. Y. Yu, Phys. Rev. D **65** (2002) 033002, hep-ph/0107261.



Total inelastic νn cross sections evaluated with the MRST 2002 NNLO PDF model modified according to Bodek–Yang prescription (solid lines) and unmodified (dashed lines).



Differences between the total neutrino cross sections for proton and neutron targets evaluated with the MRST 2002 NNLO (*left panel*) and CTEQ 5-DIS LO (*right panel*) PDF models.

7.5 Indices of refraction.

For $E_\nu \ll \min\left(m_{W,Z}^2/2m_A\right)$ and for an electroneutral nonpolarized cold medium, the q_R is energy independent. In the leading orders of the standard electroweak theory it is

$$q_R = \begin{cases} \frac{1}{2}V_0 Y_p \rho & \text{for } \alpha = e \text{ and } \beta = \mu \text{ or } \tau, \\ -\frac{1}{2}a_\tau V_0 \left(Y_p + b_\tau Y_n\right)\rho & \text{for } \alpha = \mu \text{ and } \beta = \tau, \\ \frac{1}{2}V_0 \left(Y_p - \frac{1}{2}Y_n\right)\rho & \text{for } \alpha = e \text{ and } \beta = s, \\ \frac{1}{4}V_0 Y_n \rho & \text{for } \alpha = \mu \text{ or } \tau \text{ and } \beta = s, \end{cases}$$

where

$$V_0 = \sqrt{2}G_F N_0 \simeq 7.63 \times 10^{-14} \text{ eV}$$

$$\left(L_0 = \frac{2\pi}{V_0} \simeq 1.62 \times 10^4 \text{ km} \sim D_\oplus\right),$$

$$a_\tau = \frac{3\alpha r_\tau \left[\ln(1/r_\tau) - 1\right]}{4\pi \sin^2\theta_W} \simeq 2.44 \times 10^{-5},$$

$$b_\tau = \frac{\ln(1/r_\tau) - 2/3}{\ln(1/r_\tau) - 1} \simeq 1.05,$$

α is the fine-structure constant, θ_W is the weak-mixing angle and $r_\tau = (m_\tau/m_W)^2$.

Notes:

- For an isoscalar medium the $|q_R|$ is of the same order of magnitude for any pair of flavors but $\nu_\mu - \nu_\tau$.
- For an isoscalar medium $q_R^{(\nu_\mu-\nu_\tau)}/q_R^{(\nu_e-\nu_\mu)} \approx -5 \times 10^{-5}$.
- For certain regions of a neutron-rich medium the value of $q_R^{(\nu_e-\nu_s)}$ may become vanishingly small. In this case, the one-loop radiative corrections must be taken into account.
- For very high energies the q_R have to be corrected for the gauge boson propagators and strong-interaction effects.

One can expect $|q_R|$ to be either an energy-independent or decreasing function for any pair of mixed neutrino flavors. On the other hand, there are several cases of much current interest when $|q_I|$ either increases with energy without bound (mixing between active and sterile neutrino states) or has a broad or sharp maximum (as for $\nu_\mu - \nu_\tau$ or $\overline{\nu}_e - \overline{\nu}_\mu$ mixings, respectively).

Numerical estimations suggest that for every of these cases there is an energy range in which q_R and q_I are comparable in magnitude. Since $q_R \propto \rho$ and $q_I \propto$ and are dependent upon the composition of the medium (Y_A) there may exist some more specific situations, when

$$|q_R| \sim |q_I| \sim |\Delta|$$

or even

$$|q_R| \sim |\Delta_c| \quad \text{and} \quad |q_I| \sim |\Delta_s|\,.$$

If this is the case, the refraction, absorption and mixing become interestingly superimposed.

7.6 Eigenproblem and mixing matrix in matter.

7.6.1 Eigenvalues.

The matrix $\mathbf{H}(t)$ has two complex instantaneous eigenvalues, $\varepsilon(t)$ and $-\varepsilon(t)$, with $\varepsilon = \varepsilon_R + i\varepsilon_I$ satisfying the characteristic equation

$$\varepsilon^2 = (q - q_+)\,(q - q_-)\,,$$

where

$$q_\pm = \Delta_c \pm i\Delta_s = \Delta e^{\pm 2i\theta}.$$

The solution is

$$\boxed{\begin{aligned} \varepsilon_R^2 &= \frac{1}{2}\left(\varepsilon_0^2 - q_I^2\right) + \frac{1}{2}\sqrt{\left(\varepsilon_0^2 - q_I^2\right)^2 + 4q_I^2\left(\varepsilon_0^2 - \Delta_s^2\right)}, \\ \varepsilon_I &= \frac{q_I\left(q_R - \Delta_c\right)}{\varepsilon_R} \quad \text{(provided } q_R \neq \Delta_c\text{)}\,, \end{aligned}}$$

with

$$\varepsilon_0 = \sqrt{\Delta^2 - 2\Delta_c q_R + q_R^2} \geq |\Delta_s|, \qquad \text{sign}\,(\varepsilon_R) \stackrel{\text{def}}{=} \text{sign}(\Delta) \equiv \zeta.$$

(At that choice $\varepsilon = \Delta$ for vacuum and $\varepsilon = \zeta\varepsilon_0$ if $q_I = 0$.)

In the vicinity of the MSW resonance, $q_R = q_R(t_\star) = \Delta_c$

$$\lim_{q_R \to \Delta_c \pm 0} \varepsilon_R = \Delta_s \sqrt{\max\left(1 - \Delta_I^2/\Delta_s^2, 0\right)},$$

$$\lim_{q_R \to \Delta_c \pm 0} \varepsilon_I = \pm\zeta\Delta_I \sqrt{\max\left(1 - \Delta_s^2/\Delta_I^2, 0\right)},$$

where $\Delta_I = q_I(t_\star)$. Therefore the resonance value of $|\varepsilon_R|$ (which is inversely proportional to the neutrino oscillation length in matter) is always smaller than the conventional MSW value $|\Delta_s|$ and *vanishes* if $\Delta_I^2 < \Delta_s^2$ (ε_I remains finite in this case). In neutrino transition through the region of resonance density $\rho = \rho(t_\star)$, ε_I undergoes discontinuous jump while ε_R remains continuous. The corresponding cuts in the q plane are placed outside the circle $|q| \leq |\Delta|$. If $\Delta_I^2 > \Delta_s^2$, the imaginary part of ε vanishes while the real part remains finite.

A distinctive feature of the characteristic equation is the existence of two mutually conjugate "super-resonance" points $q_\pm$ in which ε vanishes giving rise to the total degeneracy of the levels of the system (impossible in the "standard MSW" solution). Certainly, the behavior of the system in the vicinity of these points must be dramatically different from the conventional pattern.

> The "super-resonance" conditions are physically realizable for various meaningful mixing scenarios.

Some useful relations:

$$\varepsilon_R^2 = \frac{2q_I^2\left(\varepsilon_0^2-\Delta_s^2\right)}{\sqrt{\left(\varepsilon_0^2-q_I^2\right)^2+4q_I^2\left(\varepsilon_0^2-\Delta_s^2\right)}-\varepsilon_0^2+q_I^2},$$

$$\varepsilon_I = \frac{\sqrt{\left(\varepsilon_0^2-q_I^2\right)^2+4q_I^2\left(\varepsilon_0^2-\Delta_s^2\right)}-\varepsilon_0^2+q_I^2}{2q_I\left(q_R-\Delta_c\right)},$$

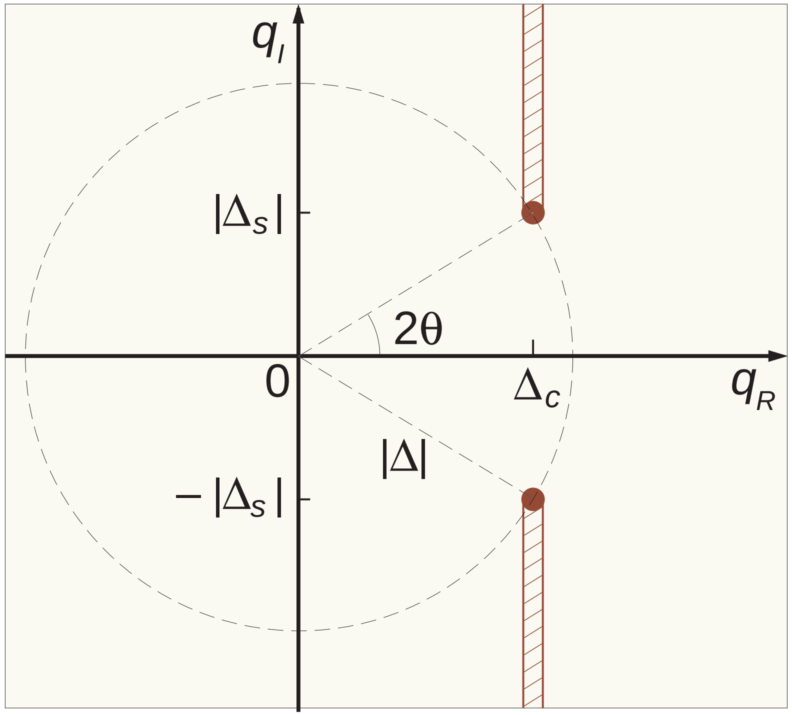
$$\frac{\partial\varepsilon_R}{\partial q_R} = \frac{\partial\varepsilon_I}{\partial q_I} = \frac{q_I\varepsilon_I+\left(q_R-\Delta_c\right)\varepsilon_R}{\varepsilon_R^2+\varepsilon_I^2},$$

$$\frac{\partial\varepsilon_I}{\partial q_R} = -\frac{\partial\varepsilon_R}{\partial q_I} = \frac{q_I\varepsilon_R-\left(q_R-\Delta_c\right)\varepsilon_I}{\varepsilon_R^2+\varepsilon_I^2},$$

$$\mathrm{Re}\left[\frac{q(t)-\Delta_c}{\varepsilon}\right] = \left(\frac{q_R-\Delta_c}{\varepsilon_R}\right)\left(\frac{\varepsilon_R^2+q_I^2}{\varepsilon_R^2+\varepsilon_I^2}\right),$$

$$\mathrm{Im}\left[\frac{q(t)-\Delta_c}{\varepsilon}\right] = \left(\frac{q_I}{\varepsilon_R}\right)\left(\frac{\varepsilon_R^2-\varepsilon_0^2+\Delta_s^2}{\varepsilon_R^2+\varepsilon_I^2}\right),$$

$$\left(q_R-\Delta_c\right)^2 = \varepsilon_0^2-\Delta_s^2.$$



Zeros and cuts of ε in the q plane for $\Delta_c > 0$. The cuts are placed outside the circle $|q| \le |\Delta|$ parallel to axis $q_R = 0$. The MSW resonance point is $(\Delta_c, 0)$ and the two "super-resonance" points are $(\Delta_c, \pm\Delta_s)$.

7.6.2 Eigenstates.

In order to simplify the solution to the eigenstate problem we'll assume that the phase trajectory $q = q(t)$ does not cross the points $q_\pm$ at any t. In non-Hermitian quantum dynamics one has to consider the two pairs of instantaneous eigenvectors $|\Psi_\pm\rangle$ and $|\overline{\Psi}_\pm\rangle$ which obey the relations

$$\mathbf{H}|\Psi_\pm\rangle = \pm\,\varepsilon|\Psi_\pm\rangle \quad \text{and} \quad \mathbf{H}^\dagger|\overline{\Psi}_\pm\rangle = \pm\,\varepsilon^*|\overline{\Psi}_\pm\rangle. \tag{25}$$

and (for $q \neq q_\pm$) form a complete biorthogonal and biorthonormal set,

$$\langle\overline{\Psi}_\pm|\Psi_\pm\rangle = 1, \quad \langle\overline{\Psi}_\pm|\Psi_\mp\rangle = 0, \quad |\Psi_+\rangle\langle\overline{\Psi}_+| + |\Psi_-\rangle\langle\overline{\Psi}_-| = \mathbf{1}.$$

Therefore, the eigenvectors are defined up to a gauge transformation

$$|\Psi_\pm\rangle \mapsto e^{if_\pm}|\Psi_\pm\rangle, \quad |\overline{\Psi}_\pm\rangle \mapsto e^{-if_\pm^*}|\overline{\Psi}_\pm\rangle,$$

with arbitrary complex functions $f_\pm(t)$ such that $\mathrm{Im}\,(f_\pm)$ vanish as $q = 0$.[a] Thus it is sufficient to find any particular solution of Eqs. (25). Taking into account that $\mathbf{H}^\dagger = \mathbf{H}^*$, we may set $|\overline{\Psi}_\pm\rangle = |\Psi_\pm^*\rangle$ and hence the eigenvectors can be found from the identity

$$\mathbf{H} = \varepsilon|\Psi_+\rangle\langle\Psi_+^*| - \varepsilon|\Psi_-\rangle\langle\Psi_-^*|.$$

[a] For our aims, the class of the gauge functions may be restricted without loss of generality by the condition $f_\pm|_{q=0} = 0$.

Setting $|\Psi_\pm\rangle = (v_\pm, \pm v_\mp)^T$ we arrive at the equations

$$v_\pm^2 = \frac{\varepsilon \pm (q - \Delta_c)}{2\varepsilon}, \quad v_+ v_- = \frac{\Delta_s}{2\varepsilon},$$

a particular solution of which can be written as

$$v_+ = \sqrt{\left|\frac{\varepsilon + q - \Delta_c}{2\varepsilon}\right|}\, e^{i(\varphi - \psi)/2},$$
$$v_- = \zeta \sqrt{\left|\frac{\varepsilon - q + \Delta_c}{2\varepsilon}\right|}\, e^{i(-\varphi - \psi)/2}.$$

where

$$\varphi = \arg(\varepsilon + q - \Delta_c) = -\arg(\varepsilon - q + \Delta_c) = \arctan\left(\frac{q_I}{\varepsilon_R}\right),$$
$$\psi = \arg(\varepsilon) = \arctan\left(\frac{\varepsilon_I}{\varepsilon_R}\right).$$

We have fixed the remaining gauge ambiguity by a comparison with the vacuum case.

7.6.3 Mixing angle in matter.

It may be sometimes useful to define the complex mixing angle in matter $\varTheta = \varTheta_R + i\varTheta_I$ by the relations

$$\sin\varTheta = v_+ \quad \text{and} \quad \cos\varTheta = v_-$$

or, equivalently,

$$\sin 2\varTheta = \frac{\Delta_s}{\varepsilon}, \quad \cos 2\varTheta = \frac{\Delta_c - q}{\varepsilon},$$

The real and imaginary parts of $\varTheta$ are found to be

$$\begin{aligned}
\mathsf{Re}(\varTheta) \equiv \varTheta_R &= \frac{1}{2}\arctan\left[\frac{(q_I - \Delta_s)\,\varepsilon_R - (q_R - \Delta_c)\,\varepsilon_I}{(q_R - \Delta_c)\,\varepsilon_R + (q_I - \Delta_s)\,\varepsilon_I}\right], \\
\mathsf{Im}(\varTheta) \equiv \varTheta_I &= \frac{1}{4}\ln\left[\frac{\varepsilon_R^2 + \varepsilon_I^2}{(q_R - \Delta_c)^2 + (q_I - \Delta_s)^2}\right].
\end{aligned}$$

$$\begin{aligned}
\cos\varTheta &= \cos\varTheta_R\cosh\varTheta_I - i\sin\varTheta_R\sinh\varTheta_I, \\
\sin\varTheta &= \sin\varTheta_R\cosh\varTheta_I + i\cos\varTheta_R\sinh\varTheta_I.
\end{aligned}$$

Having regard to the prescription for the sign of ε_R, one can verify that $\varTheta = \theta$ if $q = 0$ (vacuum case) and $\varTheta = 0$ if $\Delta_s = 0$ (no mixing or $m_1^2 = m_2^2$). It is also clear that $\varTheta$ becomes the standard MSW mixing angle with $\mathsf{Im}(\varTheta) = 0$ when $q_I = 0$ ($\varLambda_\alpha = \varLambda_\beta$).

7.6.4 Mixing matrix in matter.

In order to build up the solution to ME for the nondegenerated case one has to diagonalize the Hamiltonian. Generally a non-Hermitian matrix cannot be diagonalized by a single unitary transformation. But in our simple case this can be done by a complex orthogonal matrix (extended mixing matrix in matter)

$$\mathbf{U}_f = \mathbf{U}\exp(i\mathbf{f}),$$

where $\mathbf{f} = \mathrm{diag}\,(f_-, f_+)$ and

$$\mathbf{U} = (|\Psi_-\rangle, |\Psi_+\rangle) = \begin{pmatrix} v_- & v_+ \\ -v_+ & v_- \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}.$$

Properties of $\mathbf{U}$:

$$\mathbf{U}^T\mathbf{H}\mathbf{U} = \mathrm{diag}\,(-\varepsilon, \varepsilon),$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{1}, \quad \mathbf{U}|_{q=0} = \mathbf{V}.$$

From CE it follows that

$$\frac{\partial\varepsilon}{\partial q} = \frac{(q-\Delta_c)}{\varepsilon}$$

and thus

$$\frac{\partial v_\pm}{\partial q} = \pm\frac{\Delta_s^2 v_\mp}{2\varepsilon^2}.$$

We therefore have

$$i\mathbf{U}^T\dot{\mathbf{U}} = -\Omega\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\Omega\boldsymbol{\sigma}_2,$$

$$\Omega = \frac{\dot{q}\Delta_s}{2\varepsilon^2} = \frac{i}{4}\frac{d}{dt}\ln\left(\frac{q-q_+}{q-q_-}\right).$$

Properties of $\mathbf{U}_f$:

$$\mathbf{U}_f^T\mathbf{H}\mathbf{U}_f = \mathrm{diag}\,(-\varepsilon, \varepsilon),$$

$$\mathbf{U}_f^T\mathbf{U}_f = \mathbf{1}, \quad \mathbf{U}_f|_{q=0} = \mathbf{V},$$

$$i\mathbf{U}_f^T\dot{\mathbf{U}}_f = -\Omega e^{-i\mathbf{f}}\boldsymbol{\sigma}_2 e^{i\mathbf{f}} - \dot{\mathbf{f}}.$$

7.7 Adiabatic solution.

Formal solution to ME in the most general form:

$$\widetilde{\mathbf{S}}(t) = \mathbf{U}_f(t) \exp\left[-i\mathbf{\Phi}(t)\right] \mathbf{X}_f(t)\mathbf{U}_f^T(0). \tag{26}$$

Here $\mathbf{\Phi}(t) = \mathrm{diag}\left(-\mathit{\Phi}(t), \mathit{\Phi}(t)\right)$ and $\mathit{\Phi}(t) = \mathit{\Phi}_R(t) + i\mathit{\Phi}_I(t)$ is the complex dynamical phase, defined by

$$\mathit{\Phi}_R(t) = \int_0^t \varepsilon_R(t')dt', \quad \mathit{\Phi}_I(t) = \int_0^t \varepsilon_I(t')dt',$$

and $\mathbf{X}_f(t)$ must satisfy the equation

$$i\dot{\mathbf{X}}_f(t) = \left[\Omega(t)e^{-i\mathbf{f}(t)}\mathbf{F}(t)e^{i\mathbf{f}(t)} + \dot{\mathbf{f}}(t)\right]\mathbf{X}_f(t), \quad \mathbf{X}_f(0) = \mathbf{1},$$

where

$$\mathbf{F}(t) = e^{i\mathbf{\Phi}(t)}\boldsymbol{\sigma}_2 e^{-i\mathbf{\Phi}(t)} = \begin{pmatrix} 0 & -ie^{-2i\mathit{\Phi}(t)} \\ ie^{2i\mathit{\Phi}(t)} & 0 \end{pmatrix}.$$

It can be proved now that the right side of Eq. (26) is gauge-invariant i.e. it does not depend on the unphysical complex phases $f_\pm(t)$. This crucial fact is closely related to the absence of the Abelian topological phases in the system under consideration.

Finally, we can put $f_\pm = 0$ in Eq. (26) and the result is

$$\widetilde{\mathbf{S}}(t) = \mathbf{U}(t) \exp\left[-i\mathbf{\Phi}(t)\right] \mathbf{X}(t)\mathbf{U}^T(0), \tag{27a}$$

$$i\dot{\mathbf{X}}(t) = \Omega(t)\mathbf{F}(t)\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{1}. \tag{27b}$$

These equations, being equivalent to the ME, have nevertheless a restricted range of practical usage on account of poles and cuts as well as decaying and increasing exponents in the "Hamiltonian" $\Omega\mathbf{F}$.

7.7.1 Adiabatic theorem.

The adiabatic theorem of Hermitian quantum mechanics can almost straightforwardly be extended to ME under the requirements:

(a) the potential q is a sufficiently smooth and slow function of t;

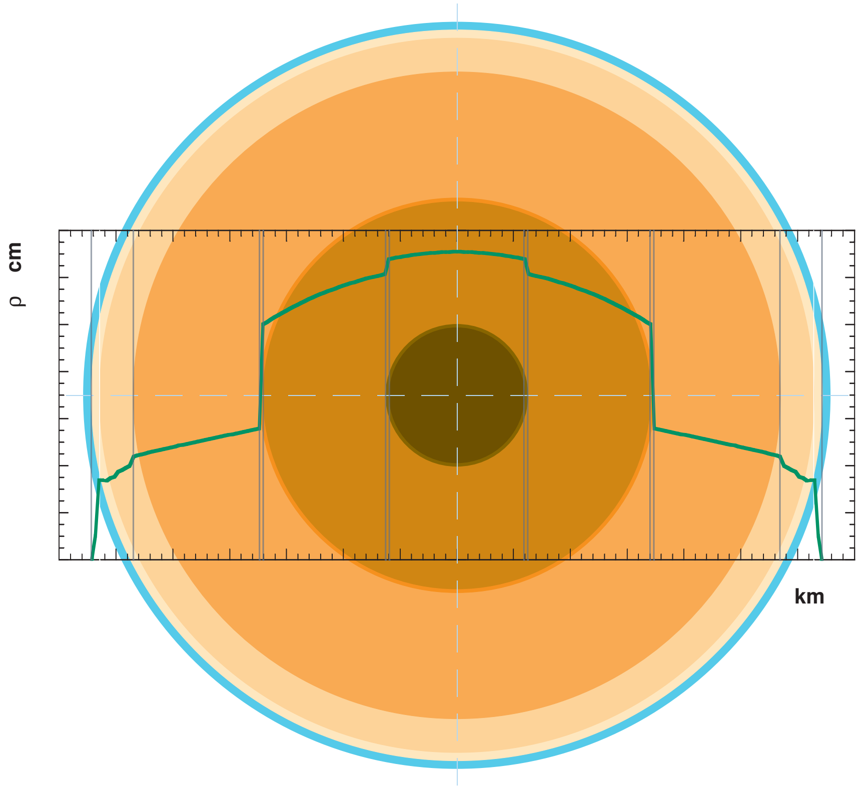
(b) the imaginary part of the dynamical phase is a bounded function i.e. $\lim_{t\to\infty} |\varPhi_I(t)|$ is finite;

(c) the phase trajectory $q = q(t)$ is placed far from the singularities for any t.

The first requirement breaks down for a condensed medium with a sharp boundary or layered structure (like the Earth). If however the requirement (a) is valid inside each layer (t_i, t_{i+1}), the problem reduces to Eqs. (27) by applying the rule

$$\widetilde{\mathbf{S}}(t) \equiv \widetilde{\mathbf{S}}(t,0) = \widetilde{\mathbf{S}}\left(t,t_n\right)\ldots\widetilde{\mathbf{S}}\left(t_2,t_1\right)\widetilde{\mathbf{S}}\left(t_1,0\right),$$

where $\widetilde{\mathbf{S}}\left(t_{i+1},t_i\right)$ is the time-evolution operator for the i-th layer.



The requirement (b) alone is not too restrictive considering that for many astrophysical objects (like stars, galactic nuclei, jets and so on) the density ρ exponentially disappears to the periphery and, on the other hand, $\varepsilon_I \to 0$ as $\rho \to 0$. In this instance, the function $\Phi_I(t)$ must be t independent for sufficiently large t. But, in the case of a steep density profile, the requirements (a) and (b) may be inconsistent. The important case of violation of the requirement (c) is the subject of a special study which is beyond the scope of this study.

It is interesting to note in this connection that, in the Hermitian case, a general adiabatic theorem has been proved without the traditional gap condition[a].

[a] J. E. Avron and A. Elgart, Commun. Math. Phys. **203** (1999) 445–467.

7.7.2 The solution.

Presume that all necessary conditions do hold for $0 \leq t \leq T$. Then, in the adiabatic limit, we can put $\Omega = 0$ in Eq. (27b). Therefore $\mathbf{X} = \mathbf{1}$ and Eq. (27a) yields

$$
\begin{aligned}
\widetilde{S}_{\alpha\alpha}(t) &= v_+(0)v_+(t)e^{-i\Phi(t)} + v_-(0)v_-(t)e^{i\Phi(t)}, \\
\widetilde{S}_{\alpha\beta}(t) &= v_-(0)v_+(t)e^{-i\Phi(t)} - v_+(0)v_-(t)e^{i\Phi(t)}, \\
\widetilde{S}_{\beta\alpha}(t) &= v_+(0)v_-(t)e^{-i\Phi(t)} - v_-(0)v_+(t)e^{i\Phi(t)}, \\
\widetilde{S}_{\beta\beta}(t) &= v_-(0)v_-(t)e^{-i\Phi(t)} + v_+(0)v_+(t)e^{i\Phi(t)},
\end{aligned}
$$

Taking into account Eq. (24) we obtain the survival and transition probabilities:

$$
\begin{aligned}
P_{\alpha\alpha}(t) &= A(t)\left\{\left[I_+^+(t)e^{\Phi_I(t)} + I_-^-(t)e^{-\Phi_I(t)}\right]^2 - I^2(t)\sin^2\left[\Phi_R(t) - \varphi_+(t)\right]\right\}, \\
P_{\alpha\beta}(t) &= A(t)\left\{\left[I_+^-(t)e^{\Phi_I(t)} - I_-^+(t)e^{-\Phi_I(t)}\right]^2 + I^2(t)\sin^2\left[\Phi_R(t) - \varphi_-(t)\right]\right\}, \\
P_{\beta\alpha}(t) &= A(t)\left\{\left[I_-^+(t)e^{\Phi_I(t)} - I_+^-(t)e^{-\Phi_I(t)}\right]^2 + I^2(t)\sin^2\left[\Phi_R(t) + \varphi_-(t)\right]\right\}, \\
P_{\beta\beta}(t) &= A(t)\left\{\left[I_-^-(t)e^{\Phi_I(t)} + I_+^+(t)e^{-\Phi_I(t)}\right]^2 - I^2(t)\sin^2\left[\Phi_R(t) + \varphi_+(t)\right]\right\},
\end{aligned}
\tag{28}
$$

where we have denoted for compactness ($\varsigma, \varsigma' = \pm$)

$$
I_\varsigma^{\varsigma'}(t) = |v_\varsigma(0)v_{\varsigma'}(t)|, \quad \varphi_\pm(t) = \frac{\varphi(0) \pm \varphi(t)}{2}, \quad I^2(t) = 4I_+^+(t)I_-^-(t) = 4I_+^-(t)I_-^+(t) = \frac{\Delta_s^2}{|\varepsilon(0)\varepsilon(t)|}.
$$

7.7.3 Limiting cases.

In the event that the conditions

$$\left|\frac{1}{\Lambda_\beta(t)}-\frac{1}{\Lambda_\alpha(t)}\right| \ll 4\varepsilon_0(t) \quad \text{and} \quad t \ll \min\left[\Lambda_\alpha(t),\Lambda_\beta(t)\right]$$

are satisfied for any $t \in [0,T]$, the formulas (28) reduce to the standard MSW adiabatic solution

$$\left.\begin{aligned} P_{\alpha\alpha}(t) &= P_{\beta\beta}(t) = \frac{1}{2}\left[1+J(t)\right]-I_0^2(t)\sin^2\left[\Phi_0(t)\right], \\ P_{\alpha\beta}(t) &= P_{\beta\alpha}(t) = \frac{1}{2}\left[1-J(t)\right]+I_0^2(t)\sin^2\left[\Phi_0(t)\right], \end{aligned}\right\} \qquad \text{(MSW)}$$

where

$$J(t)=\frac{\Delta^2-\Delta_c\left[q_R(0)+q_R(t)\right]+q_R(0)q_R(t)}{\varepsilon_0(0)\varepsilon_0(t)},$$

$$I_0^2(t)=\frac{\Delta_s^2}{\varepsilon_0(0)\varepsilon_0(t)}, \quad \Phi_0(t)=\int_0^t \varepsilon_0(t')dt'.$$

Needless to say either of the above conditions or both may be violated for sufficiently high neutrino energies and/or for thick media, resulting in radical differences between the two solutions. These differences are of obvious interest to high-energy neutrino astrophysics.

It is perhaps even more instructive to examine the distinctions between the general adiabatic solution (28) and its “classical limit”

$$\left.\begin{aligned} P_{\alpha\alpha}(t) &= \exp\left[-\int_0^t \frac{dt'}{\Lambda_\alpha(t')}\right], \quad P_{\alpha\beta}(t) = 0, \\ P_{\beta\beta}(t) &= \exp\left[-\int_0^t \frac{dt'}{\Lambda_\beta(t')}\right], \quad P_{\beta\alpha}(t) = 0, \end{aligned}\right\} \qquad (\Delta_s = 0)$$

which takes place either in the absence of mixing or for $m_1^2 = m_2^2$.

Note:

Considering that $\Omega \propto \Delta_s$, the classical limit is the exact solution to the master equation (for $\Delta_s = 0$). Therefore it can be derived directly from Eq. (23). To make certain that the adiabatic solution has correct classical limit, the following relations are useful:

$$\lim_{\Delta_s \to 0} \varepsilon(t) = \zeta\zeta_R \left[q(t) - \Delta_c\right]$$

and

$$\lim_{\Delta_s \to 0} |v_\pm(t)|^2 = \frac{1}{2}(\zeta\zeta_R \pm 1),$$

where

$$\zeta_R = \text{sign}\left[q_R(t) - \Delta_c\right].$$

7.8 Matter of constant density and composition.

In this simple case, the adiabatic approximation becomes exact and thus free from the above-mentioned conceptual difficulties. For definiteness sake we assume $\mathit{\Lambda}_\alpha < \mathit{\Lambda}_\beta$ (and thus $q_I < 0$) from here. The opposite case can be considered in a similar way. Let's denote

$$\frac{1}{\mathit{\Lambda}_\pm} = \frac{1}{2}\left(\frac{1}{\mathit{\Lambda}_\alpha} + \frac{1}{\mathit{\Lambda}_\beta}\right) \pm \frac{\xi}{2}\left(\frac{1}{\mathit{\Lambda}_\alpha} - \frac{1}{\mathit{\Lambda}_\beta}\right),$$

$$I_\pm^2 = \frac{1}{4}\left(1 + \frac{\varepsilon_0^2 + q_I^2 - \Delta_s^2}{\varepsilon_R^2 + \varepsilon_I^2}\right) \pm \frac{\xi}{2}\left(\frac{\varepsilon_R^2 + q_I^2}{\varepsilon_R^2 + \varepsilon_I^2}\right),$$

$$L = \frac{\pi}{|\varepsilon_R|} \quad \text{and} \quad \xi = \left|\frac{q_R - \Delta_c}{\varepsilon_R}\right|.$$

As is easy to see,

$$I_\pm^\pm = \begin{cases} I_\pm & \text{if} \quad \text{sign}\,(q_R - \Delta_c) = +\zeta, \\ I_\mp & \text{if} \quad \text{sign}\,(q_R - \Delta_c) = -\zeta, \end{cases}$$

$$I_+^- = I_-^+ = \sqrt{I_+ I_-} = \frac{I}{2} = \left|\frac{\Delta_s}{2\varepsilon}\right|$$

and $\text{sign}(\varphi) = -\zeta$.

By applying the above identities, the neutrino oscillation probabilities can be written as

$$P_{\alpha\alpha}(t)=\left(I_{+}e^{-t/2\Lambda_{+}}+I_{-}e^{-t/2\Lambda_{-}}\right)^{2}-I^{2}e^{-t/\Lambda}\sin^{2}\left(\frac{\pi t}{L}+|\varphi|\right),$$

$$P_{\beta\beta}(t)=\left(I_{-}e^{-t/2\Lambda_{+}}+I_{+}e^{-t/2\Lambda_{-}}\right)^{2}-I^{2}e^{-t/\Lambda}\sin^{2}\left(\frac{\pi t}{L}-|\varphi|\right),$$

$$P_{\alpha\beta}(t)=P_{\beta\alpha}(t)=\frac{1}{4}I^{2}\left(e^{-t/2\Lambda_{-}}-e^{-t/2\Lambda_{+}}\right)^{2}+I^{2}e^{-t/\Lambda}\sin^{2}\left(\frac{\pi t}{L}\right).$$

The difference between the survival probabilities for ν_α and ν_β is

$$P_{\alpha\alpha}(t)-P_{\beta\beta}(t)=-\zeta\mathrm{Re}\left(\frac{q-\Delta_{c}}{\varepsilon}\right)\left(e^{-t/2\Lambda_{-}}-e^{-t/2\Lambda_{+}}\right)$$
$$+I^{2}e^{-t/\Lambda}\sin\varphi\sin\left(\frac{2\pi t}{L}\right).$$

7.8.1 Case $|q| \gtrsim |\Delta_s|$.

Let's examine the case when Λ_+ and Λ_- are vastly different in magnitude. This will be true when $\Lambda_\beta \gg \Lambda_\alpha$ and the factor ξ is not too small. The second condition holds if q_R is away from the MSW resonance value Δ_c and the following dimensionless parameter

$$\varkappa = \frac{\Delta_s}{|q|} \approx 0.033 \times \sin 2\theta \left(\frac{\Delta m^2}{10^{-3}\ \text{eV}^2}\right)\left(\frac{100\ \text{GeV}}{E_\nu}\right)\left(\frac{V_0}{|q|}\right)$$

is sufficiently small. In fact we assume $|\varkappa| \lesssim 1$ and impose no specific restriction for the ratio q_R/q_I. This spans several possibilities:

- ⋆ small Δm^2,
- ⋆ small mixing angle,
- ⋆ high energy,
- ⋆ high matter density.

The last two possibilities are of special interest because the inequality $|\varkappa| \lesssim 1$ may be fulfilled for a wide range of the mixing parameters Δm^2 and θ by changing E_ν and/or ρ. In other words, this condition is by no means artificial or too restrictive.

After elementary while a bit tedious calculations we obtain

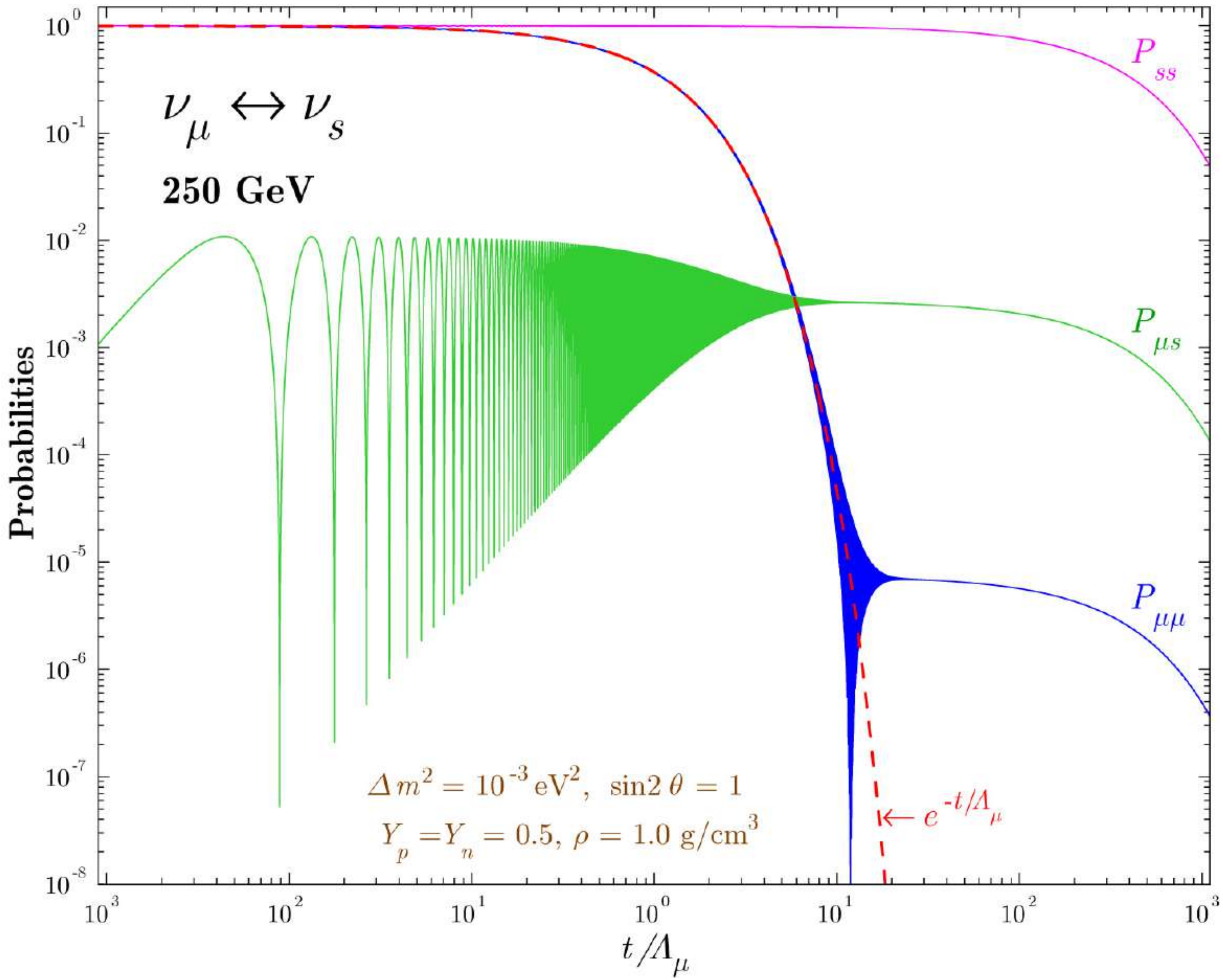
$$\xi = 1 - \frac{1}{2}\varkappa^2 + \mathcal{O}\left(\varkappa^3\right), \quad I^2 = \varkappa^2 + \mathcal{O}\left(\varkappa^3\right),$$
$$I_+ = 1 + \mathcal{O}\left(\varkappa^2\right), \quad I_- = \frac{1}{4}\varkappa^2 + \mathcal{O}\left(\varkappa^3\right);$$

$$\varLambda \ \approx 2\varLambda_\alpha, \quad \varLambda_+ \approx \left(1 + \frac{\varkappa^2}{4}\right)\varLambda_\alpha \approx \varLambda_\alpha, \quad \varLambda_- \approx \left(\frac{4}{\varkappa^2}\right)\varLambda_\alpha \gg \varLambda_\alpha.$$

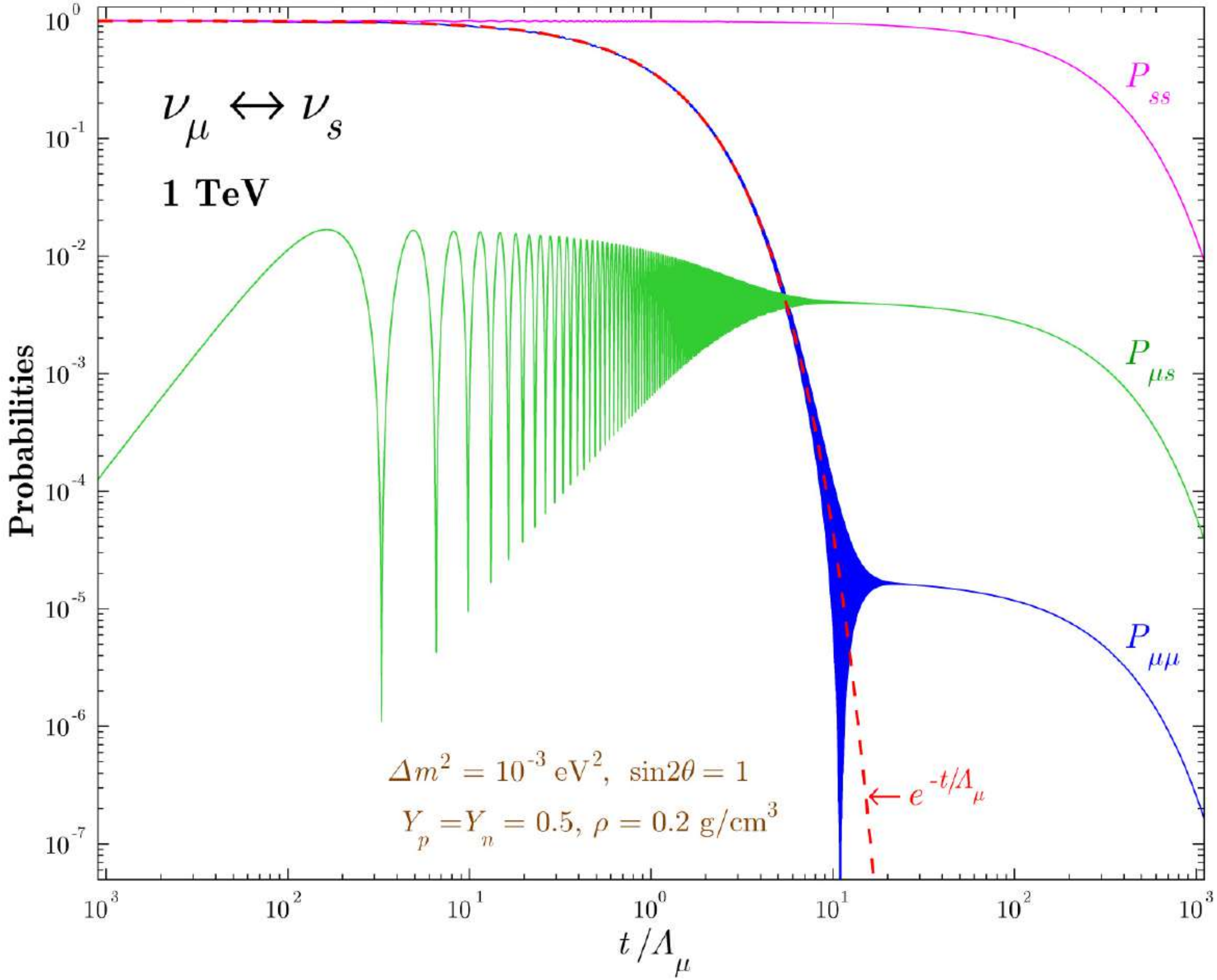
Due to the wide spread among the length/time scales $\varLambda_\pm$, $\varLambda$ and L as well as among the amplitudes $I_\pm$ and I, the regimes of neutrino oscillations are quite diverse for different ranges of variable t.

With reference to Figures in pp. 130–133, one can see a regular gradation from slow (for $t \lesssim \varLambda_\mu$) to very fast (for $t \gtrsim \varLambda_\mu$) neutrino oscillations followed by the asymptotic nonoscillatory behavior:

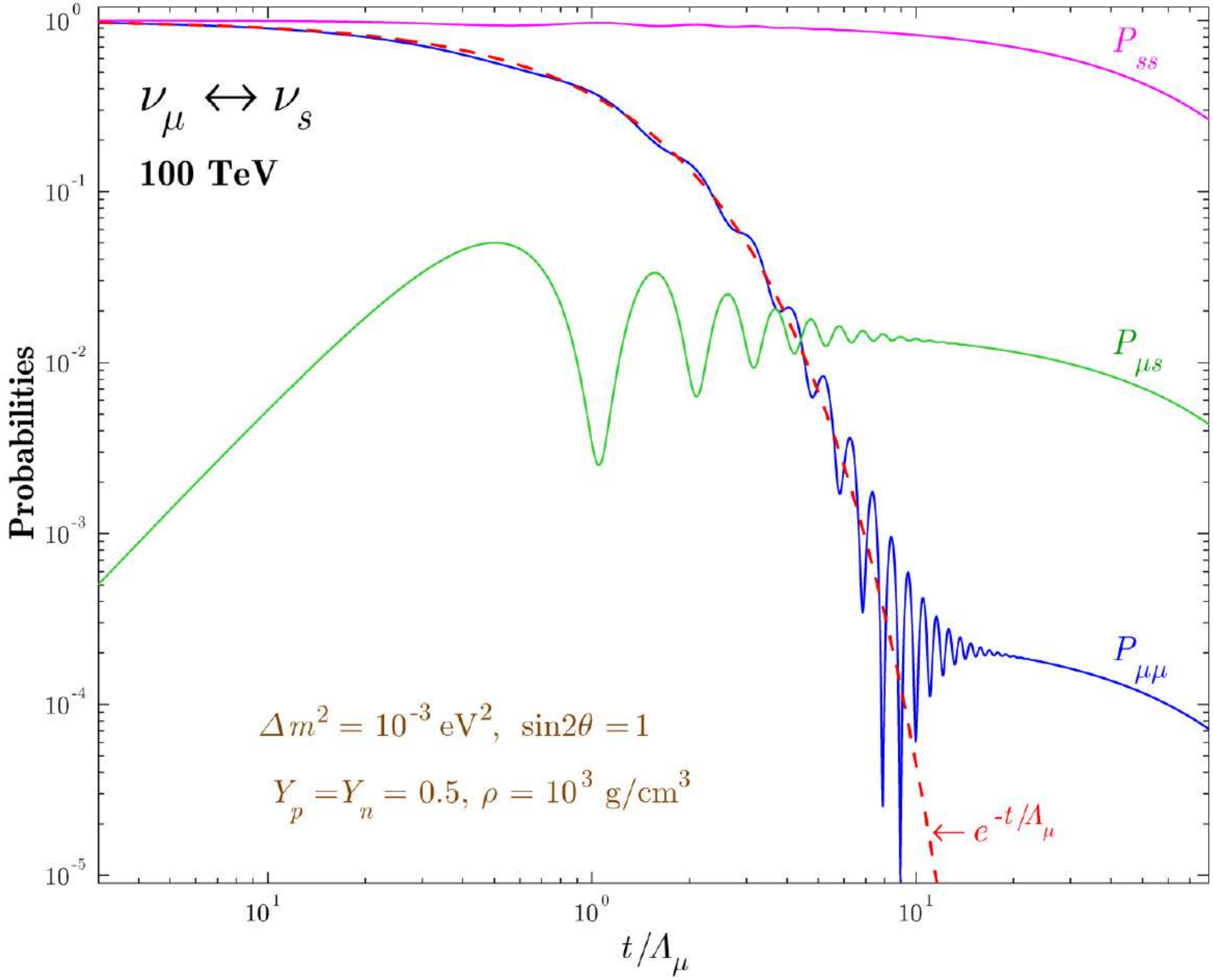
$$\boxed{\begin{aligned} &P_{\mu\mu}(t) \simeq \frac{\varkappa^4}{16}e^{-t/\varLambda_-}, \\ &P_{ss}(t) \simeq e^{-t/\varLambda_-}, \\ &P_{\mu s}(t) = P_{s\mu}(t) \simeq \frac{\varkappa^2}{4}e^{-t/\varLambda_-}. \end{aligned}}$$



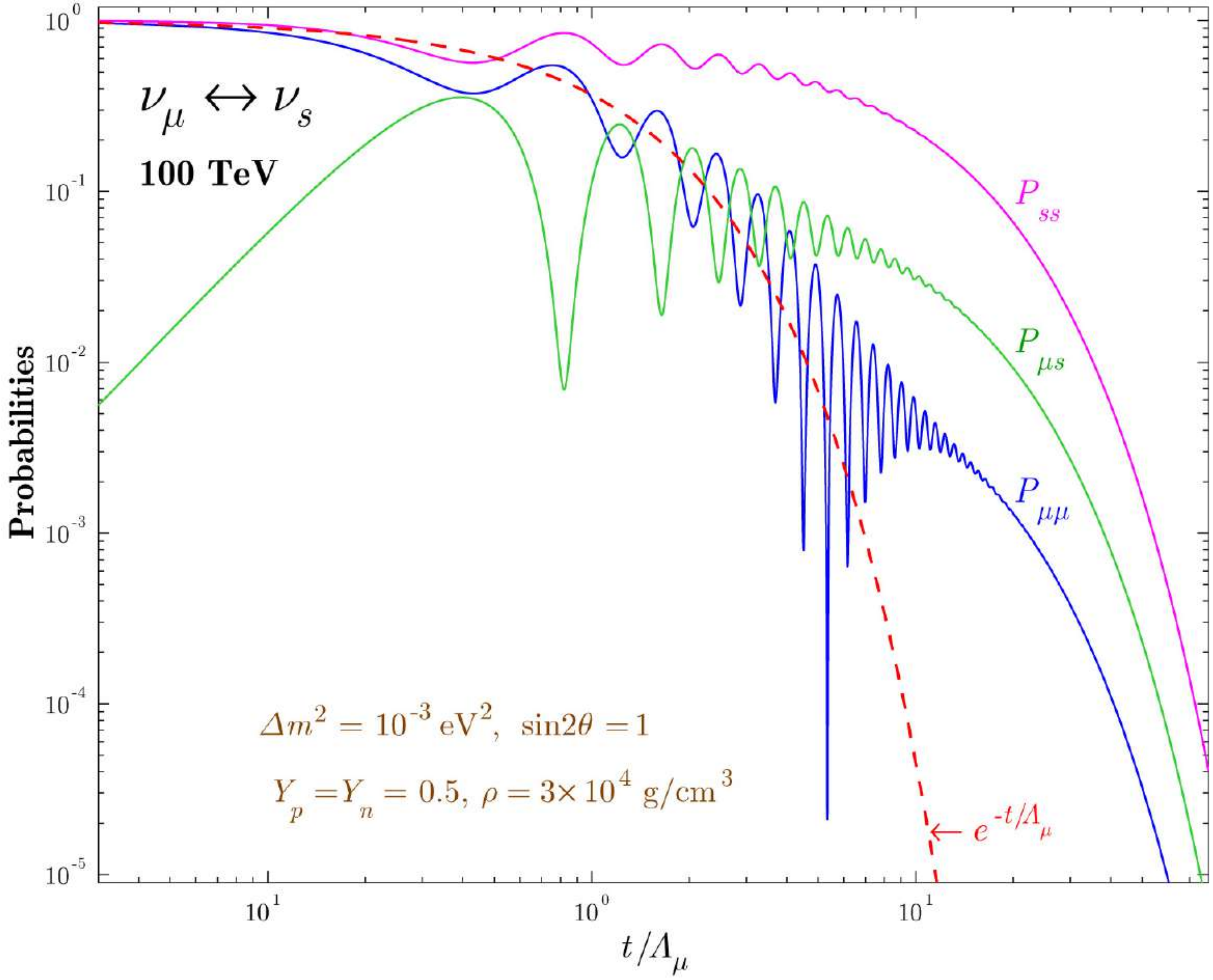
Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 250$ GeV, $\rho = 1$ g/cm^3).



Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 1000$ GeV, $\rho = 0.2$ g/cm^3).

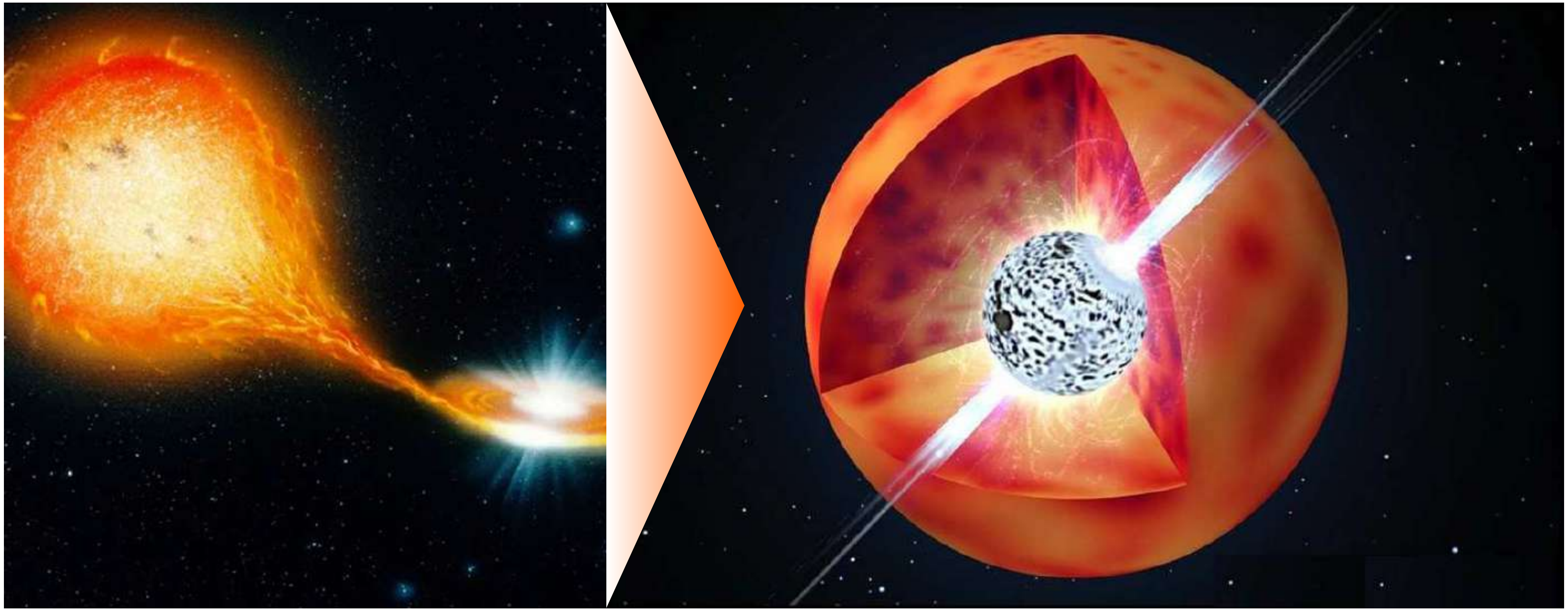


Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 100$ TeV, $\rho = 10^{-3}$ g/cm^3).



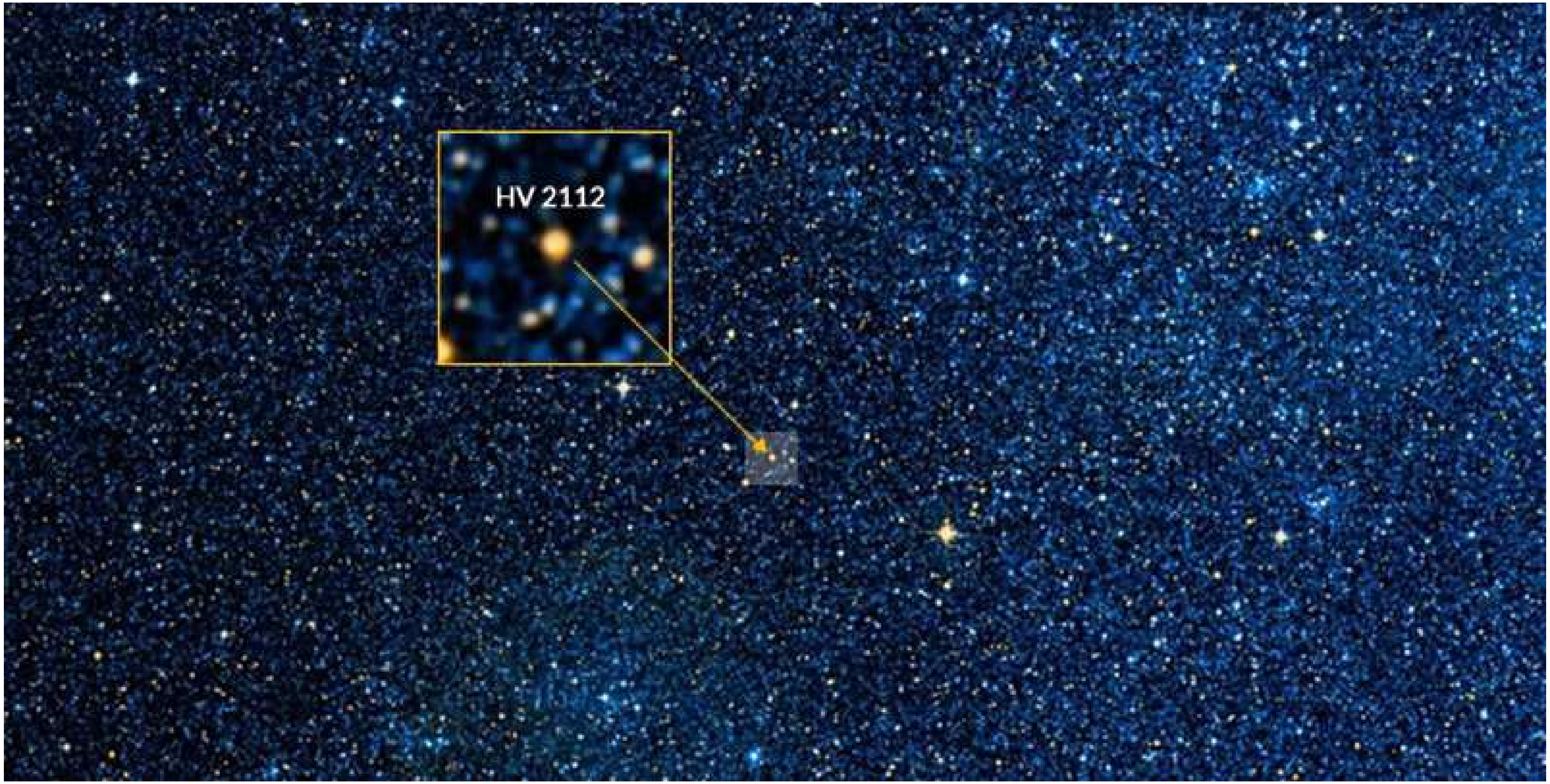
Survival and transition probabilities for $\nu_\mu \leftrightarrow \nu_s$ oscillations ($E_\nu = 100$ TeV, $\rho = 3 \times 10^{-4}$ g/cm^3).

The mechanism under discussion may be released in the Thorne–Żytkow objects (TŻO) – binaries with a neutron star submerged into a red supergiant core. Figure shows an artistic view of how a TŻO could be formed.



[See, e.g., URLs: ⟨http://astrofishki.net/universe/hv-2112-neveroyatnyj-obekt-torna-zhitkov/⟩ and ⟨http://www.decifrandoastronomia.com.br/2017/01/uma-estrela-dentro-de-outra-conheca-hv.html⟩.]

The very bright red star HV 2112 in the Small Magellanic Cloud (see next slide) could be a massive supergiant-like star with a degenerate neutron core (TŻO). With its luminosity of over $10^5 L_\odot$, it could also be a super asymptotic giant branch star (SAGB), a star with an oxygen/neon core supported by electron degeneracy and undergoing thermal pulses with third dredge up.



Both TŻO and SAGB stars are expected to be rare. Calculations performed by Ch. A. Tout *et al.* [a] indicate that HV 2112 is likely a genuine TŻO. But a much more likely explanation is that HV 2112 is an intermediate mass ($\sim 5M_\odot$) AGB star; a new TŻO candidate (HV 11417) is recently suggested.[b]

[a] Ch. A. Tout, A. N. Żytkow, R. P. Church, & H. H. B. Lau, "HV 2112, a Thorne–Żytkow object or a super asymptotic giant branch star", Mon. Not. Roy. Astron. Soc. **445** (2014) L36–L40, arXiv:1406.6064 [astro-ph.HE].

[b] E. R. Beasor, B. Davies, I. Cabrera-Ziri, & G. Hurst , "A critical re-evaluation of the Thorne–Żytkow object candidate HV 2112", arXiv:1806.07399 [astro-ph.SR].

7.8.2 Degenerate case.

The consideration must be completed for the case of degeneracy. Due to the condition $q_I < 0$, the density and composition of the "degenerate environment" are fine-tuned in such a way that

$$q = q_{-\zeta} = \Delta_c - i\,|\Delta_s|.$$

The simplest way is in coming back to the master equation. Indeed, in the limit of $q = q_{-\zeta}$, the Hamiltonian reduces to

$$\mathbf{H} = |\Delta_s| \begin{pmatrix} -i & \zeta \\ \zeta & i \end{pmatrix} \equiv |\Delta_s|\,\mathbf{h}_\zeta.$$

Considering that $\mathbf{h}_\zeta^2 = \mathbf{0}$, we promptly arrive at the solution of ME:

$$\widetilde{\mathbf{S}}(t) = \mathbf{1} - it\,|\Delta_s|\,\mathbf{h}_\zeta$$

and thus

$$\boxed{\begin{aligned} P_{\alpha\alpha}(t) &= \left(1 - |\Delta_s|\,t\right)^2 e^{-t/\mathit{\Lambda}}, \\ P_{\beta\beta}(t) &= \left(1 + |\Delta_s|\,t\right)^2 e^{-t/\mathit{\Lambda}}, \\ P_{\alpha\beta}(t) &= P_{\beta\alpha}(t) = \left(\Delta_s t\right)^2 e^{-t/\mathit{\Lambda}}. \end{aligned}}$$

Since $1/\Lambda_\beta = 1/\Lambda_\alpha - 4\,|\Delta_s|$, the necessary condition for the total degeneration is

$$4\Lambda_\alpha\,|\Delta_s| \le 1$$

and thus

$$1/\Lambda = 1/\Lambda_\alpha - 2\,|\Delta_s| \ge 2\,|\Delta_s|.$$

The equality only occurs when ν_β is sterile.

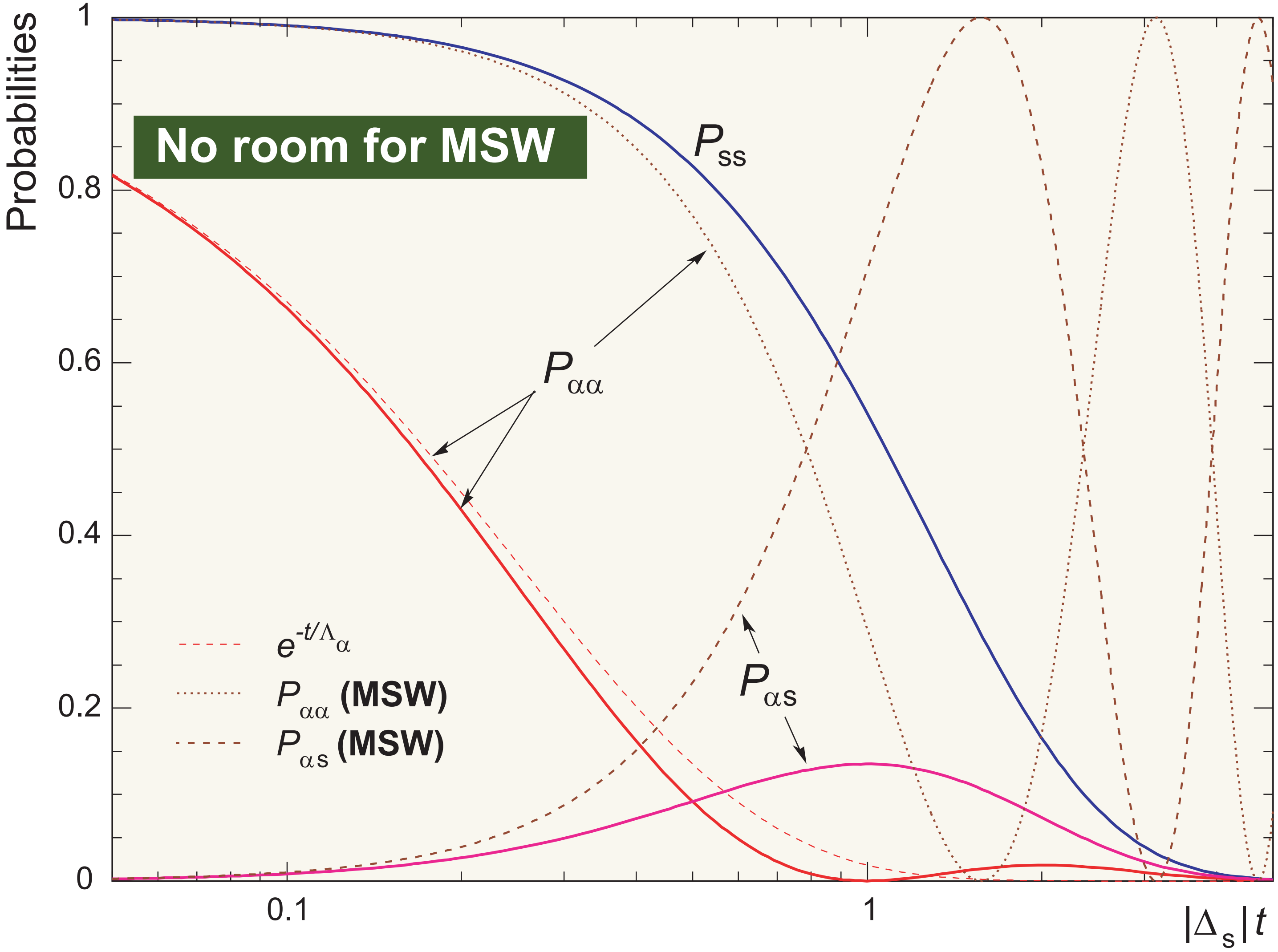
The degenerate solution must be compared with the standard MSW solution

$$\left.\begin{aligned} P_{\alpha\alpha}(t) = P_{ss}(t) &= \frac{1}{2}\left[1 + \cos\left(2\Delta_s t\right)\right], \\ P_{\alpha s}(t) = P_{s\alpha}(t) &= \frac{1}{2}\left[1 - \cos\left(2\Delta_s t\right)\right], \end{aligned}\right\} \qquad \text{(MSW)}$$

and with the classical penetration coefficient

$$\exp\left(-t/\Lambda_\alpha\right)$$

(with $1/\Lambda_\alpha$ numerically equal to $4\,|\Delta_s|$) relevant to the transport of unmixed active neutrinos through the same environment.



Survival and transition probabilities for $\nu_\alpha \leftrightarrow \nu_s$ oscillations in the case of degeneracy ($q = q_{-\zeta}$). The standard MSW probabilities (dotted and dash-dotted curves) together with the penetration coefficient for unmixed ν_α (dashed curve) are also shown.

7.9 Conclusions.

We have considered, on the basis of the MSW evolution equation with complex indices of refraction, the conjoint effects of neutrino mixing, refraction and absorption on high-energy neutrino propagation through matter. The adiabatic solution with correct asymptotics in the standard MSW and classical limits has been derived. In the general case the adiabatic behavior is very different from the conventional limiting cases.

A noteworthy example is given by the active-to-sterile neutrino mixing. It has been demonstrated that, under proper conditions, the survival probability of active neutrinos propagating through a very thick medium of constant density may become many orders of magnitude larger than it would be in the absence of mixing. The quantitative characteristics of this phenomenon are highly responsive to changes in density and composition of the medium as well as to neutrino energy and mixing parameters.

Considering a great variety of latent astrophysical sources of high-energy neutrinos, the effect may open a new window for observational neutrino astrophysics.