# Modern Trends in Mathematical Physics. I

(for pedestrians ... cyclists and drivers)

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Moscow International School of Physics 2022

JINR, Dubna, July 2022

# Mathematical Physics

Far beyond the Standard Model ....

- Officially – a small subset of theory of *differential equations*;
- Actually:
  - Either ... problems in theoretical physics with emphasis on methods of modern mathematics;
  - Or ... problems in modern mathematics coming out of the problems of theoretical physics;
- Too wide ... some restrictions:
  - (Almost) no gravity ... too hard (except for 2d);
  - 'Positive historic experience': 2d, conformal symmetry, integrals of motion etc;
  - "Exactly solvable" but ... 'realistic' ...

To work out some intuition, using exact results ... when possible ...

# Trends

- Playing with dimensions
  - Old ideas:
    - KK (dimensional reduction), string theory (dimension from anomaly cancelation);
    - Exact results in 2d (partially – in 3d).
  - New ideas:
    - KK and string theory: keeping compact dimension, dependence of its size $R$;
    - Dualities (unexpected relations) between 2d and 4d: 6d theory on $\mathbb{R}^4 \times \Sigma$;
    - Strong coupling and non-Lagrangian "field theories".

Favourite examples: 4d (better 5d) supersymmetric gauge theory and 2d conformal field theory.

# Trends

- Modern mathematics:
  - Old ideas:
    - Algebra: symmetries and representation theory;
    - Geometry: metric and topological properties ... complex structures.
  - New ideas:
    - Extention to infinite-dimensional algebras: (exact quantum anomalous dimensions from representation theory);
    - Local to global geometric properties: algebraic geometry – Riemann surfaces, Calabi-Yau manifolds etc.

Basic example: Simple mathematical models: integrable systems, cluster varieties ... differential equations (Painlevé – more than 100 years old).

# Historic background

- Non-Abelian gauge fields (1954+);
  - Asymptotic freedom (1972);
  - Instanton solutions (1975);
  - Their relation with algebraic geometry (1978);
  - Emergence from open strings (1974) and D-branes (1989);
- 2d conformal field theories (1972-1984):
  - Emergence in the context of string theory (1981);
  - Exact solutions by representation theory (1982-1984);
- Supersymmetry (1970-1974)

# Modern syntesis

- Supersymmetric gauge theory (1994);
  - Loop cancellations – cleaning the mess of perturbation theory;
  - Main role of non-perturbative contributions (e.g. instantons);
  - Dual language at strong coupling;
- Exact realization of old physical ideas (confinement as dual Meissner effect 1974-1994+);
- Exact summation over non-perturbative contributions (effective form: combinatorial sum over partitions, 2000+);
- (Unexpected?) relation to integrable systems (finite-dimensional, 1995) and 2d conformal field theories (2010+);
- ...

# Some ideas and bypasses

- Playing with dimensions;
  - 3d, 4d or 5d SUSY gauge theories, smooth connection ...;
  - Compact dimensions, 5d $= \mathbb{R}^4 \times S^1_R$, 5d in UV and 4d (with KK modes) in IR;
  - Compact euclidean time: quantum mechanics on instanton moduli space.
- Complexification:
  - quantum theory in complexified space;
  - complex manifolds (versus supersymmetry): moduli and charges;
  - CY ($\mathbb{T}^2$, K3, ...) $\to$ non-compact CY ($\mathbb{C}^3$,...) $\to$ complex curves $\Sigma$.
- Related mathematical issues:
  - From differential to difference equations $p \sim \partial_x \to \exp(Rp) \sim \exp(R\partial_x) = T_x$, sometimes much easier (!);
  - Towards quantum and affine algebras in representation theory, cluster algebras.

Try to write some formulas ...